Lecture 33: Continuous joint distributions — Chapter 5.

- For a single random variable $X$ with density function $f_X(x)$, and for any subset $B \subseteq \mathbb{R}$ (for example, $B = (a, b)$) we have

$$P(X \in B) = \int_B f(x) \, dx.$$  

- $B$ could be the union of 2 intervals: $B = (a, b) \cup (c, d)$, in which case:

$$P(X \in B) = \int_B f(x) \, dx = \int_a^b f(x) \, dx + \int_c^d f(x) \, dx.$$  

- Assume now we have two r.v.'s $X$ and $Y$ (independent or not).

* If they're discrete, they're described by a joint probability distribution:

$$P(X, Y) = P(X = x, Y = y)$$

represented graphically by a 3D histogram.

* If they're continuous, there's what we call a "joint probability density", i.e. a function of two variables:

$$z = f(x, y), \quad (x, y) \in \mathbb{R}^2,$$

whose graph is a surface, with the property that for any $B \subseteq \mathbb{R}^2$,

$$P((X, Y) \in B) = \text{Volume enclosed between } B \text{ and the graph of } f \text{ above } B.$$
This volume, as we shall see, can be computed as a "double integral" over $B$:

$$P((X,Y) \in B) = \iint_B f(x,y) \, dx \, dy$$

(which you know from Calc III if you've taken it - otherwise you'll learn how to compute it in this class. Basically, you have to compute two integrals).

Let's start "from scratch".

**Uniform distributions** (§5.1)

**Def.**: we say that $(X,Y)$ are "uniformly distributed" in a subset $D \subset \mathbb{R}^2$ of finite area if:

- $P((X,Y) \in D) = 1$ \quad (i.e. $(X,Y)$ certainly belongs to $D$)
- if $C \subset D$, then

$$P((X,Y) \in C) = \frac{\text{Area}(C)}{\text{Area}(D)}$$

**Remark**: for a single uniform continuous random variable $X$, i.e. with

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

if we define $D' = (a,b)$, then for any interval $C' = (c,d) \subset D'$ we have:

$$P(X \in C') = \frac{d-c}{b-a} = \frac{\text{Length}(C')}{\text{Length}(D')}$$

so we see that (*) is the two-dimensional counterpart of (**).
What is the density $f(x,y)$ of a uniform distribution on $D \subset \mathbb{R}^2$ of $X$ and $Y$ going to look like?

$$f(x,y) = \begin{cases} \frac{1}{\text{Area } D} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

So: \[ P((X,Y) \in D) = \text{entire volume in this "slab"} \]

\[ = \text{Area } (D) \cdot \text{height} \]

\[ = \text{Area } (D) \cdot \frac{1}{\text{Area } (D)} = 1 \]

- for $C \subset D$ ("$C$ subset of $D"$)
\[ P((X,Y) \in C) = \text{Volume above } C \text{ and under graph of } f \]

\[ = \text{Area } (C) \cdot \text{height} = \frac{\text{Area } (C)}{\text{Area } (D)} \]

which is precisely $(*)$ on the previous page.

**Theorem:** if:
- $X \sim \text{Uniform } (a,b)$, $Y \sim \text{Uniform } (c,d)$
- $X$ and $Y$ are independent

then $X,Y$ is uniformly distributed on the rectangle:

$R = (a,b) \times (c,d) = \{(x,y) | a \leq x \leq b, c \leq y \leq d \}$

"cartesian product".

So the joint probability density will be:

$$f(x,y) = \begin{cases} \frac{1}{(b-a)(d-c)} & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$
Proof: we will prove \((*)\) of page 183 for sub-rectangles of \(R\).

Consider the intervals:

\[
A \subseteq (a, b) \quad \text{and} \quad B \subseteq (c, d)
\]

by independence

\[
P((X, Y) \in A \times B) = P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B) = \left(\frac{\text{Length}(A)}{b-a}\right) \cdot \left(\frac{\text{Length}(B)}{d-c}\right) = \frac{\text{Area}(A \times B)}{\text{Area}(R)}.
\]

Remember: for discrete, independent, uniform r.v.'s \(X\) and \(Y\) on, say \(\{1, 2, 3\}\), i.e. such that

\[
P(X = k) = P(Y = k) = \frac{1}{3} \quad \text{for } k = 1, 2, 3,
\]

we also have that the joint probability distribution

\[
P(x, y) = P(X = x, Y = y)
\]

is also uniform on the discrete set of ordered pairs:

\[
\{1, 2, 3\} \times \{1, 2, 3\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}.
\]
Example: \( X, Y \) iid and Uniform \((0,1)\). Compute:

(a) \( P(Y \leq X^2) \)

(b) \( P(|Y - X| \leq \frac{1}{2}) \)

(a) We know that \( X, Y \) are uniformly distributed on the square \( D = [0,1] \times [0,1] \), so \( \text{Area}(D) = 1 \).

\[
\text{Area}(C) = \int_{0}^{1} x^2 \, dx = \frac{1}{3} \left[ x^3 \right]_0^1 = \frac{1}{3},
\]

\[
\Rightarrow P(Y \leq X^2) = \frac{\text{Area}(C)}{\text{Area}(D)} = \frac{1}{3} = \frac{1}{3}.
\]

(b) \( P\left(\left|Y - X\right| \leq \frac{1}{2}\right) = ? \)

"Distance" between \( Y \) and \( X \)

Let's solve: \( |Y - X| = \frac{1}{2} \) \( \Rightarrow \) \( Y - X = \pm \frac{1}{2} \)

we get two lines:

\[
Y = X + \frac{1}{2},
\]

\[
Y = X - \frac{1}{2},
\]

Instead of computing an integral, we use basic geometry: \( \text{Area}(C) = 1 - \text{Area}(T_1) - \text{Area}(T_2) \), i.e.

\[
\text{Area}(C) = 1 - \frac{1}{2}(\frac{1}{2})^2 - \frac{1}{2}(\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4}
\]

so \( P\left(\left|Y - X\right| \leq \frac{1}{2}\right) = \frac{\text{Area}(C)}{\text{Area}(D)} = \frac{3}{4}/1 = \frac{3}{4} \).
(c) \( P(X + Y \leq 1 \mid |Y - X| \leq \frac{1}{2}) \)

\[
P(X + Y \leq 1, |Y - X| \leq \frac{1}{2}) = \frac{P(|Y - X| \leq \frac{1}{2})}{P(|Y - X| \leq \frac{1}{2})} = \frac{P(Y \leq 1 - X)}{P_2}
\]

where \( P_2 = P(|Y - X| \leq \frac{1}{2}) = \frac{3}{4} \)

and \( P_1 = P(X + Y \leq 1, |Y - X| \leq \frac{1}{2}) = \frac{1}{2} \cdot \frac{3}{4} \)

So \( \frac{P_1}{P_2} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{1}{2} = P(X + Y \leq 1 \mid |Y - X| \leq \frac{1}{2}). \)

More in detail:

\[
\begin{align*}
\{ |Y - X| \leq \frac{1}{2} \} & \quad \{ X + Y \leq 1 \} \\
\end{align*}
\]

Intersection of the two.