Lecture 28: back to Poisson Arrival Processes.

- $\lambda > 0$ is the "ARRIVALS RATE".
- Interarrival ("Waiting") times $W_i$, $i = 1, 2, 3, \ldots$ are iid $\text{Exponential}(\lambda)$ r.v.'s; the density is
  \[ f(x) = \begin{cases} 
  \lambda e^{-\lambda x} & x > 0 \\
  0 & x < 0 
  \end{cases} \]
  \[ E(W_i) = \frac{1}{\lambda} \]
  \[ \text{Var}(W_i) = \frac{1}{\lambda^2} \]

- Arrival times: $t_1, t_2, t_3, \ldots$

Counting Process: $N(0, t] = \# \text{ of arrivals up to time } t$

- Remark: $W_i$'s are memoryless — if you have waited for a customer for $t$ minutes already, the probability of waiting another $\Delta t$ minutes is the same as the probability of having to wait $\Delta t$ minutes to start with.

We saw (but did not prove yet):

\[ N(0, t] \sim \text{Poisson}(\lambda t) \]

and in fact, if $N(s, t] = \# \text{ of arrivals in interval } (s, t]$, 
\[ N(s, t] \sim \text{Poisson}(\lambda(t-s)) \]

So:

\[ E(N(s, t]) = \lambda(t-s) \]
\[ \text{Var}(N(s, t]) = \lambda(t-s) \]
* Also: if \((s_1, t_1]\) and \((s_2, t_2]\) are disjoint time intervals, (i.e. \((s_1, t_1]\) \(\cap\) \((s_2, t_2]\) = \(\emptyset\)) then one can prove that the Poisson Random Variables \(N(s_1, t_1]\) and \(N(s_2, t_2]\) are independent. Intuitively, it follows from the fact that interarrival times in one interval (which determine the \# of arrivals in that interval) are independent from those in the other one:

\[
\text{(we will not give a rigorous proof of the independence of } N(s_1, t] \text{ and } N(s_2, t]).}
\]

* Example: Suppose that phone calls arrive at a company at a rate of \(\lambda = 3\) calls/minute, and that we model the waiting time for a call like an Exponential \((\lambda)\) random variable. Assuming that the waiting times are independent, we can model the calls as a Poisson Arrivals Process. So,

\[
P(N(s,t] = k) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}, \quad k = 0, 1, 2, \ldots
\]

- \(P(\text{no calls in first 1.5 minutes}) = P(N(0,1.5] = 0) = e^{-3(1.5)}\)
- \(P(\text{1st call arrives after 2 minutes}) = P(N(0,2] \leq 4 = \sum_{i=0}^{4} e^{-6} \frac{(6)^i}{i!}\)

\(\frac{\text{Poisson(6)}}{\text{Poisson(6)}}\)
- \(P(\text{the time between the 3rd & 4th call is more than 2 minutes}) = P(W_4 > 2) = e^{-(3)(2)} = e^{-6}\) (since \(W_4 \sim \text{Exponential(3)}\)).
- \(P(\text{2 calls in first minute} \mid 5 \text{ calls in first 2 minutes}) = P(N(0,1] = 2 \mid N(0,2] = 5) = \frac{P(N(0,1] = 2, N(0,2] = 5)}{P(N(0,2] = 5)} \frac{e^{-3/2} \frac{3^2}{2!} e^{-3/3} \frac{3^3}{3!}}{e^{-6} \frac{6^5}{5!}} = \frac{10}{25} = 0.4\)
The Gamma $(r, \lambda)$ probability density, (with ARRIVALS RATE $\lambda$)

Consider a Poisson arrivals process and the random variable

$$T_r = \text{time of the } i^{th} \text{ arrival.}$$

So \(T_r = W_1 + W_2 + W_3 + \ldots + W_r\):

it is the sum of \( r \) i.i.d \( \text{Exponential}(\lambda) \) r.v.'s.

So: \( \cdot \ E(T_r) = \frac{r}{\lambda}, \quad \text{Var}(T_r) = \frac{r}{\lambda^2}. \)

In fact, one may prove the following: the r.v. \(T_r\) has "Gamma \((r, \lambda)\)" probability density, given by

$$f(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^r}{(r-1)!} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

\( \cdot \) for \( r = 1 \) we have the usual exponential

\( \cdot \) for \( r \geq 2 \) we get densities of the following type: