Lecture 8:
- Lecture notes on line, & video recordings of lectures (link on email I sent you).
- Students interested in Actuarial P exam?
- Finish example from last time.

**Independence of 3 (or more) events.**

**Def:** three events $A, B, C$ are called independent if

1. $P(ABC) = P(A)P(B)P(C)$
2. $P(AB) = P(A)P(B)$
3. $P(AC) = P(A)P(C)$
4. $P(BC) = P(B)P(C)$

If only these three hold, then we say we have "pairwise independence" of $A, B, C$.

**Remark:** **Pairwise Independence **≠ **Independence** of $A, B, C$

* Counter-example: $\Omega = \{a, b, c, d\}$, equally likely outcomes.

Let: $A = \{a, d\}, B = \{b, d\}, C = \{c, d\}$

Then: 2), 3), 4) hold.

- $P(A) = \frac{1}{2}, P(B) = \frac{1}{2}$, so $P(A)P(B) = \frac{1}{4}$, and $P(AB) = P(\{d\}) = \frac{1}{4}$, therefore $P(AB) = P(A)P(B)$.
- The same holds for the other two pairs, $(A, C), (B, C)$.

* 1) doesn't hold.

- $P(A) = P(B) = P(C) = \frac{1}{2}$, so $P(A)P(B)P(C) = \frac{1}{8}$.
- However, $P(ABC) = P(\{d\}) = \frac{1}{4}$. 


• In general, $n$ events $A_1, A_2, \ldots, A_n$ are called independent if the probability of the intersection of any $k$ of them (with $1 \leq k \leq n$) factorizes.

• One can prove that when $A_1, \ldots, A_n$ are independent, any one of them is independent from events formed by "combining" the other ones by unions, intersections, complementations, etc. E.g.

$$P(A_1 \mid A_2 A_3 \cup A_4) = P(A_1).$$

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Chapter 2.

Assume we have $n$ repeated, independent trials of an experiment, where at the $i^{th}$ trial we can have one of the following events:

$S_i$: success, with probability $p$

$F_i$: failure, " " $q = 1 - p$

Examples:

• Tossing a coin:
  $S_i$: heads at $i^{th}$ toss, $p = \frac{1}{2}$

• Rolling two dice
  $S_i$: getting two six's, $p = \frac{1}{36}$

• Having a baby
  $S_i$: the $i^{th}$ couple has a girl, $p = \frac{487}{1000}$
If \( P(S_i) = p \) and I perform \( n \) trials, the expected number of successes is \( np \).

Let's build a tree for the number of successes \( k \) out of \( n \) trials, and compute the probability:

\[ P(k \text{ successes out of } n). \]

In this tree, we don't really care about the order of successes or failures. Note:

\[
P(1 \text{ out of } 3) = P(1 \text{ out of } 3  | 0 \text{ out of } 2)P(0 \text{ out of } 2) + P(1 \text{ out of } 3  | 1 \text{ out of } 2)P(1 \text{ out of } 2) = pq^2 + q \cdot (2pq) = 3pq^2
\]

To obtain the tree above, I just merge the nodes of a "traditional" tree that have the same number of successes (show this first).
The coefficients in the tree (*) are the numbers in "Pascal's triangle":

are denoted by \( C(n, k) \) and represent the way of choosing \( k \) elements out of \( n \) in a set (they're called "combinations of \( k \) out of \( n \) elements")

Example: if I win \( k = 2 \) times out of \( n = 3 \), I can win 1st & 2nd, 1st & 3rd, 2nd & 3rd \( \rightarrow 3 \) ways!

So \( C(3,2) = 3 \)

Theorem (from Combinatorics):

\[
C(n, k) = \frac{n!}{k!(n-k)!}, \quad \text{denoted by } \binom{n}{k}
\]

The numbers of the type \( \binom{n}{k} \) are called "binomial coefficients".

Example: \( C(3,2) = \frac{3!}{2!1!} = 3 \).

So:

\[
P(\text{k successes out of n}) = \binom{n}{k} p^k q^{n-k}
\]

"binomial distribution"
Summary so far:

- \( n \) repeated, independent trials of an experiment
  \( S_i \): success at \( i^{th} \) trial, \( P(S_i) = p \)
  \( F_i \): failure \( i^{th} \) trial, \( P(F_i) = q = 1 - p \)

- \( P_k = P( \text{k successes out of } n ) = C(n, k) \cdot p^k \cdot q^{n-k} \)
  called the "Binomial \((n, p)\) distribution", where

\[
C(n, k) = \text{# of ways I can win } k \text{ out of } n \text{ times}
\]

\[
\text{"binomial coefficients" } \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n \geq 0
\]

\[0 \leq k \leq n\]

which are exactly the numbers in Pascal's triangle:

\[
\begin{array}{cccccccccc}
& & & & & & & & & & 1 \\
& & & & & & & & & 1 & 1 \\
& & & & & & & & 1 & 2 & 1 \\
& & & & & & & 1 & 3 & 3 & 1 \\
& & & & & & 1 & 4 & 6 & 4 & 1 \\
& & & & & 1 & 5 & 10 & 10 & 5 & 1 \\
& & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
& & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
& & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
& 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
& & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 \\
& & & & \ldots & & & & & & & \\
\end{array}
\]

Note: by definition, \( n! = n(n-1)(n-2)\ldots 2 \cdot 1 \)
\( 0! = 1 \)

(show again diagram with probabilities)