**Lecture 7:**
- Quiz #1 solutions posted
- Will post solutions to HW#1 and HW#2.
- **Question:** who's interested in the Actuarial P exam?
- HW#2 due now!

**Clarifications:**
- Two events $A$ and $B$ are "mutually exclusive" if $AB = \emptyset$, i.e., if $A$ and $B$ are disjoint sets.
  
  In this case:
  \[
  P(A \cup B) = P(A) + P(B).
  \]

- Mutually exclusive $\neq$ independent!
  If $A$ and $B$ are mutually exclusive, then
  \[
  P(AB) = 0 \neq P(A)P(B) \quad \text{(unless } P(A) = 0 \text{ or } P(B) = 0).\]

  If $A$ and $B$ are independent, then
  \[
  P(AB) = P(A)P(B) \neq 0 \quad \text{(unless } P(A) = 0 \text{ or } P(B) = 0).\]

- When we talk about conditional probabilities given an event $B$ (with $P(B) > 0$), we often use the terms:

  "PRIOR probability of $A$" for $P(A)$

  "POSTERIOR probability of $A$ (given $B$)" for $P(A|B)$.

Now, back to sequential events...
Example: a proportion \( p \in [0, 1] \) of the faces of a die are painted white.

- a proportion \( q = 1 - p \) of the faces of the same die are painted black.

The die is rolled \( n \) times (with \( n \) arbitrary).

\( W_i \): at the \( i \)th roll, we get white.

\( B_i \): at the \( i \)th roll, we get black.

This type of tree is called "binary tree", because each node has exactly two branches.

Note: the rolls are independent, so, for example,

\[
P(B_2 | B_1) = P(B_2) = q
\]

Q.: Let's compute:

\[
P(\text{it takes 3 rolls or less to get white}) = P(W_1 \cup B_1W_2 \cup B_1B_2W_3) = P(W_1) + P(B_1W_2) + P(B_1B_2W_3)
\]

\[
= P(W_1) + P(B_1)P(W_2) + P(B_1)P(B_2)P(W_3)
\]

\[
= p + qP + q^2P = (1 + q + q^2)p = (*)
\]

Remark: \((*) = (1 + q + q^2)(1 - q) = 1 + q + q^2 - q - q^2 - q^3 = 1 - q^3 = 1 - P(B_1)P(B_2)P(B_3) = 1 - P(B_1B_2B_3) = P((B_1B_2B_3)^c)\), and that's because

"it takes three rolls or less to get white" = "I do not get three consecutive blacks".
Q: \( P( \text{it takes 4 rolls or more to get white} ) \)
\[ = 1 - P( \text{it takes 3 rolls or less to get white} ) \]
\[ = 1 - (1 - q^3) = q^3 = P(B_1B_2B_3) \]

Q: \( P( \text{getting white for the first time at the } k^{th} \text{ roll} ) \)
\[ = P(B_1B_2...B_{k-1}W_k) = P(B_1)P(B_2)...P(B_{k-1})P(W_k) \]
\[ = q^{k-1}p \]

This is the so-called "GEOMETRIC DISTRIBUTION".

Remark:
if \( 0 \leq q < 1 \)
then \( \lim_{k \to \infty} q^k = 0 \),
i.e.
\( P(B_1B_2...B_k) \to 0 \) as \( k \to \infty \).

Q: \( P( \text{getting white in } k \text{ rolls or less} ) \)
\[ = P(W_1 \cup B_1W_2 \cup B_1B_2W_3 \cup ... \cup B_1B_2...B_{k-1}W_k) \]
\[ = p + qp + q^2p + ... + q^{k-1}p = (1 + q + ... + q^{k-1})p \]
\[ = (1 + q + ... + q^{k-1})(1 - q) = 1 + q + ... + q^{k-1} - q - q^2 - ... - q^k = 1 - q^k \]

Q: \( P( \text{getting white at any point in time} ) = \) (increases with \( k \))
\[ = \lim_{k \to \infty} (1 - q^k) = 1 - \lim_{k \to \infty} q^k = 1 \text{ if } q < 1. \]
Another way to see it:

\[ P(\text{getting white at any point in time}) = p + qp + qp^2 + qp^3 + \ldots \]

\[ = \sum_{k=0}^{\infty} pq^k = p \sum_{k=0}^{\infty} q^k = p \frac{1}{1-q} = (1-q) \frac{1}{1-q} = 1 \]

"geometric series"! (See Calculus II).

So, with probability 1, if I keep rolling I will eventually get white.

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**Probability of a flush** (flush: 5 cards of the same suit)

Pick 5 cards randomly out of 52.

| Suits:     | H: hearts ♠ | D: diamonds ♦ | S: spades ♣ | C: clubs ♠♣♠ ♣♣♠ |

\[ H_i: i^{th} \text{ card is hearts} \]

\[
P(\text{Hearts flush}) = P(H_1 H_2 H_3 H_4 H_5) = P(H_1) P(H_2 | H_1) P(H_3 | H_1 H_2) P(H_4 | H_1 H_2 H_3) P(H_5 | H_1 H_2 H_3 H_4)
\]

\[
= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}
\]

\[
P(\text{flush}) = P(H_1 H_2 H_3 H_4 H_5 \cup D_1 \ldots D_5 \cup S_1 \ldots S_5 \cup C_1 \ldots C_5) = P(H_1 \ldots H_5) + P(D_1 \ldots D_5) + P(S_1 \ldots S_5) + P(C_1 \ldots C_5)
\]

\[
= 4 \cdot P(H_1 \ldots H_5) = 0.00198 \quad \text{(less than 0.2%)}.
\]
The birthday problem. Given a class of \( n \) students, what is \( P(\text{two or more students share the same birthday}) \)?

1. Intuitively, the larger the class, the closer this is to 1.
2. First of all, order the students from 1 to \( n \).
3. Ask the first student for his/her birthday.
4. Then ask the 2nd, 3rd... and stop when you find that two have the same birthday.
5. Let: \( R_j \): "the process stops (with a repeat birthday) at \( j \)th student" \( 2 \leq j \leq n \)
6. \( D_j \): "the first \( j \) birthdays are all different"
7. \( B_n \): "two or more students have the same birthday" out of \( n \)

After asking the first student:

\[
\begin{align*}
\frac{1}{365} & R_2 \\
\frac{364}{365} & D_2 \\
\frac{363}{365} & R_3 \\
\frac{362}{365} & D_3 \\
\frac{361}{365} & R_4 \\
\frac{360}{365} & D_4 \\
& \vdots \\
\frac{n-1}{365} & R_n \\
\frac{365-(n-1)}{365} & D_{n-1}
\end{align*}
\]

Remark:
- if the first 3 birthdays are different, so are the first two. So \( D_3 \subseteq D_2 \), and \( D_2 D_3 = D_3 \).
- therefore \( P(D_4 | D_2 D_3) = P(D_4 | D_3) = \frac{362}{365} \)
- similarly, if \( i < j \) then \( D_j \subseteq D_i \), and
- \( D_2 D_3 D_4 ... D_j = D_j \). Also, \( P(D_{j+1} | D_j D_{j-1} ... D_2) = \frac{365-j}{365} \) Now,
- \( P(B_n) = P(R_2 \cup R_3 \cup ... \cup R_n) = \)
- can also be written as:
- \( P(B_n) = P(D_n^c) = 1 - P(D_n) \); \( P(D_n) = ? \)
Since \( D_{n} \subseteq D_i \), for all \( i \),
\[
D_2 D_3 D_4 \ldots D_n = D_n
\]
so
\[
P(D_n) = P(D_2 D_3 D_4 \ldots D_n)
\]
\[
= \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-(n-1)}{365}
\]
\[
= \left(1 - \frac{1}{365}\right) \left(1 - \frac{2}{365}\right) \ldots \left(1 - \frac{n-1}{365}\right)
\]
And \( P(B_n) = 1 - P(D_n) \) (**).
Proof of Gauss's formula: Define: 

\[ S_n = \sum_{i=1}^{n} i = 1+2+3+\ldots+n \]

Theorem: \[ S_n = \frac{n(n+1)}{2} \]

\begin{align*}
1 & \ 2 \ 3 \ 4 \ \ldots \ n-3 \ n-2 \ n-1 \ n & \text{sum} = S_n \\
\text{invert order} \rightarrow & \ n \ n-1 \ n-2 \ n-3 \ 4 \ 3 \ 2 \ 1 & \text{sum} = S_n
\end{align*}

\[ \sum_{i=1}^{n-1} i = S_n - n = \frac{n(n+1)}{2} - n = \frac{n^2 + n - 2n}{2} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2} \]

Which is what we used in the birthday problem.

\(\text{He proved it when he was 10 years old!}\)