Lecture 2. Last time we saw the three ingredients of an experiment:

1) The outcome space, which is a set $\Omega$.
   E.g.: if I roll a die, $\Omega = \{1,2,3,4,5,6\}$.
   ($\Omega$ could be finite, countable, or even uncountable
   like $\Omega = [a,b]$ or $\Omega = \mathbb{R}$).
   The elements of $\Omega$ are called the 'outcomes'.

2) Events = subsets of $\Omega$.
   E.g.: if I roll a die, $A = \{2,4,6\}$ or $B = \{1\}$ are events.
   Remark: $\emptyset$ and $\Omega$ are also events.

When the outcomes are "equally likely",

3) The probability of an event $A$: $P(A) = \frac{\#(A)}{\#(\Omega)}$.

We also introduced the odds of an event $A$:

$$Odds(A) = \frac{\#(A)}{\#(\Omega) - \#(A)} = \frac{P(A)}{1 - P(A)}.$$

E.g.: if $\Omega = \{1,2,3,4,5,6\}$, and $A = \{1,6\}$,
then $P(A) = \frac{2}{6} = \frac{1}{3}$, $Odds(A) = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$,

i.e. $A$ is twice more likely of not happening than of happening.
Gambling: payoff odds of an event $A$:

$$r_{\text{pay}}(A) = \text{dollars won for every dollar bet on event } A$$

E.g.: "the payoff odds of $A$ are 10 to 1" we write $r_{\text{pay}}(A) = 10$, and it means that, for every $\$1$ I bet on $A$:
- If $A$ happens, I get back $\$1 + \$10 = \$11$.
- If $A$ does not happen, I get back $\$0$.

- We will see this later in the course, but a game is **FAIR if**

$$r_{\text{pay}}(A) = \frac{1}{\text{Odds}(A)}.$$

If odds of $A$ happening are low, then payoff must be high.

- $r_{\text{pay}}(A) < \frac{1}{\text{Odds}(A)}$, then the game is **unfair**.

**Example**: in the Nevada Roulette there are 38 numbers:

$\Omega = \{1, 2, ..., 36, 0, 00\}$.

$A = \{6\}$. $r_{\text{pay}}(A) = 35$ ("35 to 1")

If you bet $\$1$ and you win, you get $\$36$ back.

However:

$P(A) = \frac{1}{38}$

so $\text{Odds}(A) = \frac{\frac{1}{38}}{1 - \frac{1}{38}} = \frac{1}{37}$, and $1 - \frac{1}{37} = 37 > 35 = r_{\text{pay}}(A)$

so the game is **UNFAIR**.
**Frequentist Interpretation of Probability.**

If $A$ is an event (a subset of the outcome space) in a given experiment, what is the interpretation of $P(A)$?

- The intuitive meaning is the "degree of certainty" that $A$ will occur. Can we make this more precise?
- Consider an outcome set $\Omega$ and an event $A \subseteq \Omega$.

If an experiment (such as tossing a coin, or rolling a die) can be repeated, each repetition is called a "trial".

Assuming that the outcome in each trial is independent, $P(A)$ may be interpreted as the proportion of times $A$ occurs. More precisely, we define:

$$n = \text{the trial number;}$$

$$n(A) = \text{the # of trials } A \text{ happens, out of the first } n.$$  

The "relative frequency" of $A$ is the proportion of the trials in which $A$ occurs:

$$f_n(A) = \frac{n(A)}{n}$$  

- This is not the number of outcomes in $A$, $\#(A)$, but the # of trials in which $A$ occurs.

E.g.: Let’s toss a coin: $\Omega = \{H, T\}$. Let’s take $A = \{H\}$.

Let’s conduct repeated trials of the experiment:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{outcome}$</th>
<th>$n(A)$</th>
<th>$f_n(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>H</td>
<td>1</td>
<td>$\frac{1}{1}$</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>H</td>
<td>2</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>4</td>
<td>H</td>
<td>3</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>5</td>
<td>T</td>
<td>3</td>
<td>$\frac{3}{5}$</td>
</tr>
<tr>
<td>6</td>
<td>T</td>
<td>3</td>
<td>$\frac{3}{6}$</td>
</tr>
<tr>
<td>7</td>
<td>H</td>
<td>4</td>
<td>$\frac{3}{7}$</td>
</tr>
<tr>
<td>8</td>
<td>H</td>
<td>5</td>
<td>$\frac{4}{8}$</td>
</tr>
<tr>
<td>9</td>
<td>H</td>
<td>6</td>
<td>$\frac{5}{9}$</td>
</tr>
<tr>
<td>10</td>
<td>H</td>
<td>6</td>
<td>$\frac{6}{10}$</td>
</tr>
</tbody>
</table>
We have that, as $n$ increases, $f_n(A)$ oscillates around and in fact gets closer to $\frac{1}{2}$.

- In the frequentist interpretation of probability,
\[ P(A) \approx \frac{1}{n} \] for large $n$,

i.e. the proportion of trials in which $A$ occurs, for a large number of trials.

- As mathematicians, we are tempted to write
\[ (*) \quad P(A) = \lim_{n \to \infty} f_n(A), \]

however $f_n(A)$ is not really a function of $n$, because it depends on the specific sequence of trials!

- If I repeat the sequence of trials, I'll get a different sequence $f_n(A)$. So, $f_n(A)$ is a "RANDOM FUNCTION" of the number $n$.

- However, it will still converge to $\frac{1}{2}$ (show different graphs).

- The limit (*) can be given a precise mathematical meaning, with "the law of large numbers" (which we will see in 3-4 weeks).
This frequentist approach can be used to measure the probability of events.

E.g.: Experiment: having a baby! What is the gender?

\[ \Omega = \{ \text{boy, girl} \} \]

\[ A = \{ \text{boy} \} \]

\[ P(A) = ? \]

We can't repeat the experiment thousands of times with the same mother, but we can consider all the births in the US as repeated, independent trials of the same experiment.

It turns out: \( P(\{\text{boy}\}) \approx 0.513 \) (roughly 105 boys each 100 girls), measured as the proportion of boys of all newborns.

Remark: the frequentist approach to the interpretation of \( P(A) \) does not work when independent repeated trials of an experiment cannot be performed. For example:

- "The probability of going to war with Iraq is 80%" makes no sense, from the frequentist point of view (the same historical situation never repeats itself). The statement is an opinion.

- "The chance of surviving a specific surgery is 80%". Again, every patient is different, and every surgeon is different too. 80% is the historical survival rate, but each case is different.