Systemic Risk: the Effect of Market Confidence

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Abstract

In a crisis, when faced with insolvency, banks can sell stock in a dilutive offering in the stock market and borrow money in order to raise funds. We propose a simple model to find the maximum amount of new funds the banks can raise in this way. To do this we incorporate market confidence of the bank together with market confidence of all the other banks into the overnight borrowing rate. Additionally, for a given cash shortfall, we find the optimal mix of borrowing and stock selling. We show the existence and uniqueness of Nash equilibrium strategy for all these problems. We then calibrate this model to market data and conduct an empirical study to assess whether the current financial system is safer than it was before the last financial crisis.

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1 Introduction

Traditionally risk management has considered how the risks affect a particular institution, while ignoring how these risks might affect the financial system as a whole. In contrast, systemic risk considers how this risk spreads throughout the financial system through the interactions of the banks in the system. Such a spread of defaults is also known as contagion. It can occur through both local and global connections, e.g., contractual obligations and impacts on borrowing rates and liquidity respectively. Such a systemic event caused the last financial crisis during which the entire financial network was threatened with insolvency. It became apparent that a good model and understanding of systemic risk are vital.

Eisenberg and Noe (2001) introduced a network framework which models default contagion through contractual obligations and found the resulting equilibrium payments. The model has been widely used by both regulators and academics e.g. Anand et al. (2015); Halaj and Kok (2015); Boss et al. (2004); Elsinger et al. (2013); upper (2011); Gai et al. (2011); Bardoscia et al. (2017a). Many other extensions have been considered, a survey of which can be found in e.g. Weber and Weske (2017); Staum (2013); Hüsler (2015). In this paper, we incorporate market confidence...
into the ability to cover shortfall and show that the cash reserve on the balance sheet offer an incomplete picture of achievable reserves of a bank. We explain how the lack of confidence spreads in the bank system when facing liquidity problem.

Liquidity, capital and the connectivity of the financial network are highly related to systemic risk. Freixas et al. (2000) investigated the ability of the banking system to withstand financial contagion and examined the too-big-to-fail policy. Diamond and Rajan (2005) showed that financial contagion shrinks the common pool of liquidity, creating or exacerbating aggregate liquidity shortage, which in turn leads to additional contagion and even a total meltdown of the system. Cifuentes et al. (2005) stated that capital requirements can cause abnormal effects between portfolios valuations and systemic resilience. Gaspar et al. (2004) studied systemic risk from the perspective of monetary policy. Their model showed how operational framework of monetary policy can affect the elasticity of supply of funds by banks throughout the reserve maintenance period. Nier et al. (2007) found that low level of equity increases the number of contagious defaults and that contagion is non-monotonic on degree of connectivity. Liquidity and confidence have also been noted in Glasserman and Young (2015) as examples of systemic contagion that may generate big losses.

The financial crisis also pointed out the need of finding a signal that can help the banks to monitor the financial situation and survive during the crisis. Gray et al. (2007) used contingent claims analysis to measure systemic risk in a structural approach. Acharya et al. (2017) used systemic expected shortfall as a measure of systemic risk. Additional empirical research including Huang et al. (2009) studied expected credit losses by data of CDS and stock return correlations. De Jonghe (2010) estimated tail betas of European financial firms to measure systemic risk. Billio et al. (2012) measured systemic risk through Granger causality test. Confidence is a natural way to measure this risk, and credit rating and overnight rates were found to be important predictors of bank’s health. Mencia (2009) stated that rating is a good factor to predict liquidity shock, with lower rating banks more likely to receive liquidity shocks. Akram and Christophersen (2010) argued that there are evidence that credit ratings, liquidity demand and supply have stronger effects on interest rate since the start of the current financial crisis, and there is a relatively large variation in actual overnight interest rates over time and across banks. Their analysis has suggested that such variation, can be partly ascribed to bank’s characteristics. In particular, domestic banks, which are considered to be ‘too big to fail’ and ‘too connected to fail’ are able to borrow at relatively lower rates than other banks.

Barucca et al. (2016) introduced a network valuation model, where the value of the financial obligations between banks can be computed and the recovery rates on those can be derived. Bardoscia et al. (2017) allowed for the contagion to spread before the actual default happen through exposure updates due to changes in default probabilities. Heijmans et al. (2013) studied how the changes in the monetary policy framework have affected the overnight money market lending rate for Dutch segment of the Europe area during tranquil and crisis times. Their results showed that modifications of the monetary policy framework in 2004 decreased the volatility of the rate, which increased again at the beginning of the crisis.

The size, connectivity degree and other such factors of the banks are all realizations of the market confidence. Arinaminpathy et al. (2012) emphasized importance of market confidence. They pointed out that the importance of relatively large, well-connected banks in system stability scales is proportionately larger than their size: the impact of their collapse arise not only from their connectivity, but also from their effect on confidence in the system. Bräuning and Fecht (2016) investigated the effect that lending relationship has on the availability and pricing of interbank liquidity. Their result implied relationship lenders are more likely to provide cheaper liquidity to their closest borrowers, and particularly opaque borrowers obtain liquidity at lower rates when borrowing from their relationship lenders.
There has also been a focus on the liquidity problem and the overnight interest rate is believed to be a good proxy for liquidity issues. \textcite{Furfine2000} formalized the relationship between bank payment activities and the federal funds rate. In their model, uncertainty of bank reserve balances was an increasing function of payment volume and bank payment activities were affected by the maintenance-period patterns of federal funds rate. \textcite{Furfine2001} provided evidence that banks are effective monitors of their peers by showing interest rate paid on federal funds transaction reflects the differences in credit risk across borrowers. They also argued that the size and relative importance in the funds market of the trading institutions can affect the rates charged for overnight borrowing. \textcite{Iori2006} argued that an interbank market lets participants pool the risk of liquidity shortage, but also creates the potential for one bank’s crisis to propagate through the system.

In general, when the banks need to raise funds, they can do it by borrowing. Popular alternatives, include fire sales and dilutive offering, i.e. selling additional equity in the company. Depending on the amount of funds that need to be raised the banks can find the optimal combination of these means. For example, \textcite{Bichuch2019} discussed fire sale and borrowing. In this paper, we assume the means to raise funds are through borrowing and stock sales, by issuing new shares or through a dilutive offering. It is assumed that both actions lower the market’s confidence in the bank and increase the overnight rate, as shown e.g. by \textcite{Rochet1996} and \textcite{Elyasiani2014}, and therefore leads to an increase in borrowing costs. Additionally, such transactions affect the entire financial sector, thus decreasing the confidence in all the banks. We model the overnight interest rate of each bank as a function of: 1) the number of shares of stock this bank and other banks sell, 2) the amount of money borrowed by this bank and the amounts borrowed by all other banks. This sets up the optimization problem as a non-cooperate concave n-person game. We show the existence and uniqueness of Nash equilibrium for this optimization problem, and establish a slightly different sufficient condition for the uniqueness of equilibrium than \textcite{Rosen1965} [Theorem 6]. Additionally, we calibrate this model with real data to find the maximum amount Citi and JP Morgan Chase banks could raise over the last decades. We then compare how these amounts changed before, during and after the financial crisis.

The rest of the paper is organized in the following way. In Section 2 we introduce the model and study the maximal amount the banks can raise by only selling/issuing stocks, with no borrowing. In Section 3 we add borrowing into consideration, and find the optimal strategy for the banks to raise cash by borrowing and selling stocks. In Section 4 we study the optimal strategy for the banks to recover their cash shortfall while minimizing their financing cost. In Section 5 we calibrate the model parameters to real data and perform the empirical analysis. Finally, we conclude and summarize the main results in Section 6.

2 Optimal Strategy Raising Funds by Selling Stock

Consider a system with $m \geq 2$ banks that suffer a shortfall or alternatively face a stress test scenario where the banks need to raise money. Our goal is to see how (the lack of) confidence spreads throughout the system, or alternatively, how the liquidity contagion evolves. We first aim to find the maximum amount of funds these $m$ banks can raise in this scenario. In this paper, we assume that the banks can raise funds in the two ways – 1) they can sell stocks; 2) they can borrow money. They are can also do both at the same time. In this section, we consider the case, when banks can only sell stocks, and when borrowing is not allowed. We then relax this assumption in Section 3. The proposed model is static, so even though raising funds in the stock market takes time, we assume that stock sale is done instantaneously and simultaneously with borrowing (when
allowed).

Denote \( s_j, j \in \{1, ..., m\} \) to be the number of stocks shares that bank \( i \) sells, and let \( s = (s_1, s_2, ..., s_m) \), and \( s_{-j} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_m) \). Such a sale affects all banks, therefore bank \( j \in \{1, ..., m\} \) stock price drops from \( p_j(0, 0) \) to \( p_j(s_j, s_{-j}) \), after such sale. This transaction also changes the book value \( B_j \) of the bank, by \( B_j(s_j, s_{-j}) - B_j(0, 0) = p_j(s_j, s_{-j})s_j \). Finally, denote \( C_j \), to be bank’s \( j \) market capitalization. Therefore, after the sale transaction, the market capitalization becomes \( C_j(s_j, s_{-j}) = p_j(s_j, s_{-j})(N_j + s_j) \), where \( N_j \) is the number of outstanding shares before this transaction.

We will use bank’s \( j \) price to book ratio \( \frac{C_j}{B_j} \) as a proxy for the market’s confidence in the bank. In turn, we assume that the bank’s overnight borrowing rate \( r_j \) depends on the confidence in the bank, and therefore it is a function of the price-to-book ratio. Additionally, we will assume that \( r_j \) depends also on the confidence in all other banks in the system. Since both \( B_j, C_j, j \in \{1, ..., m\} \) are functions of \( s \), for convenience and consistency with \( p_j \), we write \( r_j(s_j, s_{-j}) \) as well.

Notice that high price-to-book ratio, suggests high confidence in a bank, resulting in low overnight interest rate, and vice-versa. It follows that when the bank sells its shares, its book-to-price ratio, (the reverse ratio) will increase. This implies that its overnight rate will increase after this transaction as well. Moreover, since such a transaction will decrease the supply of funds available for borrowing in the market and indicate a potential for an increase in systemic risk for the whole banking sector, the overnight rates of all other banks will increase as well.

Together these are the two mechanism that are assumed in this paper that liquidity crisis, or the lack of confidence spreads between banks. As bank \( j \) sells stocks, the stock price for all other banks decrease, dragging down their proceed from stock sales. Moreover, the transaction of bank \( j \) also changes its price-to-book ratio, which also raises the short-term borrowing rates for the other banks, which in turn may cause a liquidity crisis.

Banks utilize the overnight lending market is to cover its short-term liabilities. These accounted for 44% of all banks debt in 2007, and still account for nearly a quarter of all debt in 2018 [Onaran 2018]. Denote \( L_j \) to be the size of bank \( j \)’s overnight loan. Therefore, the funds bank \( j \) can raise from a stock sale, are given by

\[
    u_j(s_j, s_{-j}) = p_j(s_j, s_{-j})s_j - L_j(r_j(s_j, s_{-j}) - r_j(0, 0)),
\]

where the first term accounts for the cash raised by stock sale, and the second term is the loss on the short-term loan due to the rate increase. We assume that \( u_j: [0, S_j] \times \prod_{i=1,i\neq j}^m [0, S_i] \to \mathbb{R} \), where \( S_i, i = 1, 2, ..., m \) is maximal number of shares bank \( i \) can sell/issue, and refer to \( u_j \) as the objective function of bank \( j \). Denote the mapping \( s_j^* : \prod_{i=1,i\neq j}^m [0, S_i] \to [0, S_j] \) to be the solution of the optimization problem for bank \( j \) given \( s_{-j} \), i.e.

\[
    s_j^*(s_{-j}) = \arg \max_{s \in [0, S_j]} u_j(s, s_{-j}). \tag{1}
\]

Additionally, for any given \( s \in \prod_{i=1}^m [0, S_i] \), define a mapping \( S^* : \prod_{i=1}^m [0, S_i] \to \prod_{i=1}^m [0, S_i] \) as

\[
    S^*(s) = (s_1^*(s_{-1}), ..., s_m^*(s_{-m}))^T. \tag{2}
\]

The following assumptions are then needed to establish existence and uniqueness of Nash equilibrium:

**Assumption 2.1.** 1. The stock price function of bank \( j \) given by \( p_j: [0, S_j] \times \prod_{i=1,i\neq j}^m [0, S_i] \to \mathbb{R}_+ \) is a twice differentiable, convex function, for any \( j \in 1, 2, ..., m \). Moreover, for any given \( s_{-j} \in \prod_{i=1,i\neq j}^m [0, S_i] \), \( s \mapsto p_j(s, s_{-j}) \) is an decreasing function in \( s \), and \( s \mapsto p_j(s, s_{-j})s \) is an...
increasing concave function in s satisfying 
\[ \frac{\partial p_j(S, s_{-j})}{\partial s_j} S_j + p_j(S_j, s_{-j}) > 0 \] 
and \[ \frac{\partial^2 p_j(s, s_{-j})}{\partial s_j^2}(N_j + s) + 2 \frac{\partial p_j(s, s_{-j})}{\partial s_j} < 0, \] \( s \in [0, S_j] \) for every \( j = 1, 2, ..., m \).

2. The overnight rate \( r_j: [0, S_j] \times \prod_{i=1, i \neq j}^m [0, S_i] \rightarrow \mathbb{R}_+ \) is twice differentiable, convex function satisfying \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) > 0 \) for any \( s_{-j} \in \prod_{i=1, i \neq j}^m [0, S_i] \).

**Remark 2.2.** Under Assumption 2.1, for any given \( j = 1, 2, ..., m \) and \( s_{-j} \in \prod_{i=1, i \neq j}^m [0, S_i] \) the function \( s \mapsto u_j(s, s_{-j}) \) is a twice differentiable concave function in \( s \).

**Example 2.3.** Inverse demand functions satisfying Assumption 2.1 include:

1. Linear price impact: \( p_j(s_j, s_{-j}) = p_j(0, 0)(1 - a_j s_j - \epsilon \sum_{i=1, i \neq j}^m s_i) \), with \( 0 < a_j < \frac{1}{2S_j}, 0 < \epsilon \ll a_j \).

2. Exponential price impact: \( p_j(s_j, s_{-j}) = p_j(0, 0)e^{-a_j s_j - \epsilon \sum_{i=1, i \neq j}^m s_i} \), with \( 0 < a_j < \frac{1}{2S_j}, 0 < \epsilon \ll a_j \).

**Theorem 2.4.** (Existence of Nash Equilibrium) Under Assumptions 2.1, there exists Nash equilibrium to the optimization problem (1) denoted by \( s^{**} = (s^*_1(s^{**}_{-1}), ..., s^*_m(s^{**}_{-m})) \in \prod_{j=1}^m [0, S_j] \).

**Proof.** For any \( j = 1, 2, ..., m \) and \( s_{-j} \in \prod_{i=1, i \neq j}^m [0, S_i] \), the first order optimality condition for an interior maximizer of the objective function \( u_j(s, s_{-j}) \) is

\[
p_j(s, s_{-j}) + s \frac{\partial p_j(s, s_{-j})}{\partial s_j} - L_j \frac{\partial r_j}{\partial s_j} = 0,
\]

which is equivalent to

\[
\left( p_j(s, s_{-j}) + s \frac{\partial p_j(s, s_{-j})}{\partial s_j} \right) \left( \frac{\partial r_j}{\partial s_j} \right)^{-1} = L_j,
\]

due to the fact that \( \frac{\partial r_j}{\partial s_j}(s, s_{-j}) > 0 \), for \( s \in [0, S_j] \), in turn which follows from Assumptions 2.1. Moreover, according to the same assumption, the derivative of the left hand side of (3),

\[
\left( \frac{\partial^2 p_j(s, s_{-j})}{\partial s_j^2} + 2 \frac{\partial p_j(s, s_{-j})}{\partial s_j} \right) \frac{\partial r_j}{\partial s_j} + \frac{\partial^2 r_j}{\partial s_j^2},
\]

is always negative, which implies the left hand side of (3) is strictly decreasing with respect to \( s \in [0, S_j] \).

Then, for this \( s_{-j} \in \prod_{i=1, i \neq j}^m [0, S_i] \), we have the following:

- If \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) \leq \frac{p_j(0, s_{-j})}{L_j} \), then if the solution to (3) exists on \( s \in [0, S_j] \), it is unique. Denote it \( s^*_j(s_{-j}) \). Otherwise, if there is no solution on \( [0, S_j] \), set \( s^*_j(s_{-j}) = \infty \).

- If \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) > \frac{p_j(0, s_{-j})}{L_j} \), there is no solution to (3) on \( s \in [0, S_j] \). In this case, selling any stocks is not a optimal. Thus set \( s^*_j(s_{-j}) = 0 \).

In summary, each bank \( j \) chooses to sell \( s^*_j(s_{-j}) \) shares provided that other banks are selling \( s_{-j} \) shares of their stock which is given by

\[
s^*_j(s_{-j}) = \begin{cases} 0 & : \frac{\partial r_j}{\partial s_j}(0, s_{-j}) \leq \frac{p_j(0, s_{-j})}{L_j}, \\ s^0_j(s_{-j}) \land S_j & : \text{otherwise}. \end{cases}
\]
In fact, \( s^* \): \( \prod_{i=1,i \neq j}^m [0, S_i] \to [0, S_j] \) is continuous in each of its components. Indeed, \( s^*_j \) is continuous as a function of \( s_{-j} \) in the region where \( \frac{\partial r_i}{\partial s_j}(0, s_{-j}) < \frac{p_j(0,s_{-j})}{L_j} \) and \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) > \frac{p_j(0,s_{-j})}{L_j} \). As \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) \uparrow \frac{p_j(0,s_{-j})}{L_j} \) implies \( s^*_j(s_{-j}) \downarrow 0 \), we get the continuity at point \( \frac{\partial r_j}{\partial s_j}(0, s_{-j}) = \frac{p_j(0,s_{-j})}{L_j} \). Hence, the mapping \( S^* \) defined in (2) is also continuous.

Therefore, by Brouwer’s fixed point Theorem there exists an equilibrium stock selling strategy given by \( s^{**} = (s^*_1(s^{**}_1), \ldots, s^*_m(s^{**}_m)) \in \prod_{i=1}^m [0, S_i] \).

We now turn to focus on the uniqueness of the Nash equilibrium strategy.

Define \( F(s) = \left( \frac{\partial u_1}{\partial s_1}(s), \ldots, \frac{\partial u_m}{\partial s_m}(s) \right)^T \), and let \( J(s) \) be the Jacobian matrix of the mapping \( F \), given by \( (J(s))_{k,j} = \frac{\partial^2 u_k}{\partial s_j \partial s_i}(s), j, k = 1, \ldots, m \).

**Lemma 2.5.** Under Assumptions 2.1, suppose also that the Jacobian matrix \( J(s) \) is strictly diagonally dominant for on \( \prod_{i=1}^m [0, S_i] \), i.e.

\[
\left| \frac{\partial^2 u_i}{\partial s^2_i} \right| > \sum_{k=1, k \neq i}^m \left| \frac{\partial^2 u_i}{\partial s_k \partial s_i} \right|, \quad i = 1, \ldots, m,
\]

then there exists a unique Nash equilibrium.

**Proof.** According to Theorem 2.4, there exists a fixed point of the mapping \( S^* \). This is a Nash equilibrium. We now show uniqueness of the fixed point. Assume by contradiction that there are two distinct points fixed points \( s, \bar{s} \in \prod_{i=1}^m [0, S_i] \). The goal is to show that in \( L_\infty \) norm

\[
\|S^*(s) - S^*(\bar{s})\|_\infty < \|s - \bar{s}\|_\infty,
\]

thereby reaching a contradiction.

Let \( j \in \{1, 2, \ldots, m\} \). There are three possible cases for the relationships between the values of \( \frac{\partial u_j}{\partial s_j} \left( s^*_j(s_{-j}), s_{-j} \right) \) and \( \frac{\partial u_j}{\partial s_j} \left( \bar{s}^*_j(\bar{s}_{-j}), \bar{s}_{-j} \right) \).

In case when \( \frac{\partial u_j}{\partial s_j} \left( s^*_j(s_{-j}), s_{-j} \right) = \frac{\partial u_j}{\partial s_j} \left( \bar{s}^*_j(\bar{s}_{-j}), \bar{s}_{-j} \right) \), then from [Gabay and Moulin 1978][Theorem 4.1], we get that \( |s^*_j(s_{-j}) - \bar{s}^*_j(\bar{s}_{-j})| < \|s - \bar{s}\|_\infty \).

Next, consider the case when

\[
\frac{\partial u_j}{\partial s_j} \left( s^*_j(s_{-j}), s_{-j} \right) < \frac{\partial u_j}{\partial s_j} \left( \bar{s}^*_j(\bar{s}_{-j}), \bar{s}_{-j} \right).
\]

In this case, either \( s^*_j(s_{-j}) \) or \( \bar{s}^*_j(\bar{s}_{-j}) \) is not an internal point of the interval \([0, S_j]\) (or both). Thus either \( s^*_j(s_{-j}) \in \{0, S_j\} \) or \( \bar{s}^*_j(\bar{s}_{-j}) \in \{0, S_j\} \). If \( s^*_j(s_{-j}) = \bar{s}^*_j(\bar{s}_{-j}) \), then we trivially have that \( 0 = |s^*_j(s_{-j}) - \bar{s}^*_j(\bar{s}_{-j})| < \|s - \bar{s}\|_\infty \). Otherwise, \( s^*_j(s_{-j}) \neq \bar{s}^*_j(\bar{s}_{-j}) \). Then, from the optimality of the mapping \( s^*_j \), it follows that either \( s^*_j(s_{-j}) = 0, s^*_j(\bar{s}_{-j}) \in (0, S_j), \) or \( s^*_j(s_{-j}) = 0, s^*_j(\bar{s}_{-j}) = S_j \). Therefore in either case \( s^*_j(s_{-j}) < s^*_j(\bar{s}_{-j}) \).

Define \( x = (s^*_j(s_{-j}), s_{-j}) \) and \( \bar{x} = (\bar{s}^*_j(\bar{s}_{-j}), \bar{s}_{-j}) \). Let \( f(t) = \frac{\partial u_j(x + t(\bar{x} - x))}{\partial s_j} \). According to Lagrange Mean Value Theorem, there exists \( t_0 \in (0, 1) \) such that

\[
f'(t_0) = \sum_{i=1, i \neq j}^m \frac{\partial^2 u_j(y)}{\partial s_j \partial s_i}(\bar{s}_i - s_i) + \frac{\partial^2 u_j(y)}{\partial s^2_j}(s^*_j(s_{-j}) - \bar{s}^*_j(s_{-j})) > 0,
\]
where \( y = x + t_0 (\tilde{x} - x) \). From which we conclude that

\[
- \frac{\partial^2 u_j (y)}{\partial s_j^2} \left( s_j^* (\tilde{s} - s_j) - s_j^* (s - s_j) \right) < \sum_{i=1, i \neq j}^{m} \frac{\partial^2 u_j (y)}{\partial s_j \partial s_i} (\tilde{s}_i - s_i) .
\] (7)

By concavity of function \( u_j \), together with the strictly diagonal dominance condition \([4]\), we get that \( \frac{\partial^2 u_j (y)}{\partial s_j^2} < 0 \) is strictly negative. Recall that \( s_j^* (s - s_j) < s_j^* (\tilde{s} - s_j) \), thus it follows that the left hand side of \((7)\) is positive. Therefore

\[
\left| \frac{\partial^2 u_j (y)}{\partial s_j^2} \left( s_j^* (\tilde{s} - s_j) - s_j^* (s - s_j) \right) \right| < \sum_{i=1, i \neq j}^{m} \left| \frac{\partial^2 u_j (y)}{\partial s_j \partial s_i} (\tilde{s}_i - s_i) \right| \leq \sum_{i=1, i \neq j}^{m} \left| \frac{\partial^2 u_j (y)}{\partial s_j \partial s_i} (\tilde{s}_i - s_i) \right| .
\]

where the last inequality follows from the strictly diagonal dominance property. We conclude that

\[
|s_j^* (\tilde{s} - s_j) - s_j^* (s - s_j)| < \|\tilde{s} - s\|_{\infty} .
\]

The case of reverse inequality in \([6]\) can be treated by renaming \( s \) and \( \tilde{s} \).

We have showed that \( |s_j^* (\tilde{s} - s_j) - s_j^* (\tilde{s} - s_j)| < \|\tilde{s} - s\|_{\infty} \) for arbitrary \( j \in 1, 2, .., m \) and thus \([5]\) follows, and we reach the desired contradiction.

The strict diagonally dominant condition of \( J(s) \) is not very intuitive, therefore we formulate the uniqueness theorem with slightly more financially intuitive condition

**Theorem 2.6.** (Uniqueness of Nash Equilibrium) In addition to Assumptions \([2, 7]\), suppose also that for each \( j = 1, 2, .., m \),

\[
-s_j \frac{\partial^2 p_j}{\partial s_j^2} (s) + L_j \frac{\partial^2 r_j}{\partial s_j^2} (s) \geq \sum_{i=1, i \neq j}^{m} s_j \frac{\partial^2 p_j}{\partial s_i \partial s_j} (s) - L_j \frac{\partial^2 r_j}{\partial s_i \partial s_j} (s) ,
\] (8)

then the Nash equilibrium \( s^{**} \) is unique.

**Proof.** According to Lemma \([2, 5]\), it suffices to prove the Jacobian matrix \( J(s) \) is diagonal strictly dominant. We calculate that

\[
J(s) = 2 \begin{bmatrix}
\frac{\partial p_1}{\partial s_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{\partial p_m}{\partial s_m}
\end{bmatrix} + \begin{bmatrix}
s_1 \frac{\partial^2 p_1}{\partial s_1^2} - L_1 \frac{\partial^2 r_1}{\partial s_1^2} & \cdots & s_1 \frac{\partial^2 p_1}{\partial s_m \partial s_1} - L_1 \frac{\partial^2 r_1}{\partial s_m \partial s_1} \\
\vdots & \ddots & \vdots \\
s_m \frac{\partial^2 p_m}{\partial s_1 \partial s_m} - L_m \frac{\partial^2 r_m}{\partial s_1 \partial s_m} & \cdots & s_m \frac{\partial^2 p_m}{\partial s_m^2} - L_m \frac{\partial^2 r_m}{\partial s_m^2}
\end{bmatrix} .
\]

Condition \([8]\) ensures that the second matrix on the right hand side is a diagonal dominant matrix with negative diagonal elements. Since \( \frac{\partial p_j}{\partial s_j} (s) < 0 \), the sum of the two matrix on the right hand side is strictly diagonal dominant. Hence, by Lemma \([2, 5]\) the equilibrium point is unique. \(\square\)
accommodate this change, we change the definitions introduced in the previous section by adding a debt, does not affect the total equity directly, according to Committee and Board (2000). To other banks), and raise the interest rate for all the banks in the system. This is due again because including itself, and also raise the short-term interest rate, again for all banks. Now, if a bank takes completes the contagion circle. Previously, a bank selling stock, will lower the stock price for all banks, Our goal is still to find the maximum amount of funds the banks in the system can raise. This com-

\[ \text{Remark 2.7. The intuition behind the condition } \] is that it ensures that the effect of bank’s own transactions on its marginal cost is greater than the combined effects of all other banks.

\[ \text{Note also that the condition of Lemma 2.5 is very closely related to the sufficient condition of } \]

\[ \text{Rosen (1965)/Theorem 6}. \] The latter being that \( J(s) + J(s)^T \) is positive definite. However, they are not equivalent as there exists strictly (row) diagonally dominant matrices \( J \), not satisfying the positiveness condition of \( J(s) + J(s)^T \). Finally, the simple self-containing proof is itself noteworthy.

### 3 Optimal Strategy Selling Stock and Borrowing

We now relax the original assumption and allow banks to borrow cash, in addition to selling stock. Our goal is still to find the maximum amount of funds the banks in the system can raise. This completes the contagion circle. Previously, a bank selling stock, will lower the stock price for all banks, including itself, and also raise the short-term interest rate, again for all banks. Now, if a bank takes on additional debt, we assume that it will both lower the stock price for the bank (and possibly for other banks), and raise the interest rate for all the banks in the system. This is due again because of the decreasing confidence in this, and (possibly) other banks. Though note that borrowing, as a debt, does not affect the total equity directly, according to Committee and Board (2000). To accommodate this change, we change the definitions introduced in the previous section by adding the dependency on debt \( d \), in addition to their dependencies on stock sale \( s \). Denote \((s,d) = (s_1, ..., s_m, d_1, ..., d_m)\), and let \((s_{-i}, d_{-i}) = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_m, d_1, ..., d_{i-1}, d_{i+1}, ..., d_m)\), \(i = 1, ..., m\), where \(d_j\) denotes the amount of money the bank \(j \in \{1, ..., m\}\) borrows. Then bank \(j\) stock price function becomes \(p_j(s_j, d_j, s_{-j}, d_{-j})\). After the joint stock sale and borrowing transaction the banks book value and market capitalization become \(B_j(s_j, d_j, s_{-j}, d_{-j}) = B_j(0, 0, 0, 0) + p_j(s_j, d_j, s_{-j}, d_{-j})s_j\) and \(C_j(s_j, d_j, s_{-j}, d_{-j}) = p_j(s_j, d_j, s_{-j}, d_{-j})(N_j+s_j)\). Similarly, the overnight interest rate now becomes \(r_j = r_j(s_j, d_j, s_{-j}, d_{-j})\).

Hence, for any fixed \(j \in 1, 2, ..., m\), the amount of funds the bank \(j\) can raise as a function of \((s_j, d_j)\) given the actions of other banks \((s_{-j}, d_{-j})\) is

\[ v_j(s_j, d_j, s_{-j}, d_{-j}) = s_jp_j(s_j, d_j, s_{-j}, d_{-j}) + d_j - L_j (r_j(s_j, d_j, s_{-j}, d_{-j}) - r_j(0, 0, 0, 0)) \]

The effect of borrowing on stock price is complicated. When a large number of banks face shortfall, the common sense is the stock prices decrease as borrowing increases, whereas the marginal costs increases These facts suggest that the stock price function \(p_j\) is a decreasing concave function with respect to \(d \in \prod_{i=1}^m [0, D_i]\), for any given \(s \in \prod_{i=1}^m [0, S_i]\), where \(D_i\) is the maximum amount of cash the bank \(j \in \{1, ..., m\}\) can borrow.

#### Example 3.1. Inverse demand functions satisfying Assumption 2.7 include:

1. Linear Inverse Demand: \(p_j(s_j, d_j, s_{-j}, d_{-j}) = p_j(0, 0, 0, 0)(1 - a_j s_j - b_j d_j^2 - \epsilon_1 \sum_{k=1, k\neq j}^m s_k - \epsilon_2 \sum_{k=1, k\neq j}^m d_k^2)\), with \(0 < a_j < \frac{1}{s_j}, 0 < b_j < \frac{1}{d_j^2}, 0 < \epsilon_1 \ll a_j, 0 < \epsilon_2 \ll b_j\).

2. Exponential Inverse Demand: \(p_j(s_j, d_j, s_{-j}, d_{-j}) = p_j(0, 0, 0, 0)\exp(-a_j s_j - b_j d_j^2 - \epsilon_1 \sum_{k=1, k\neq j}^m s_k - \epsilon_2 \sum_{k=1, k\neq j}^m d_k^2)\), with \(0 < a_j < \frac{1}{s_j}, 0 < b_j < \frac{1}{d_j^2}, 0 < \epsilon_1 \ll a_j, 0 < \epsilon_2 \ll b_j\).

Define a mapping \(\bar{F} : \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i] \to \mathbb{R}^{2m}\) as \(\bar{F}(s, d) = (\frac{\partial m}{\partial s_1}, \frac{\partial m}{\partial d_1}, ..., \frac{\partial m}{\partial s_m}, \frac{\partial m}{\partial d_m})^T\), and let \(\bar{J}(s, d)\) be the Jacobian matrix of \(\bar{F}\).

Our goal is to investigate the existence and uniqueness of Nash equilibrium in this system. The following assumption is then needed to give a definite answer.
Assumption 3.2. 1. For each \( j = 1, \ldots, m \), given \((s_{-j}, d_{-j}) \in \prod_{i=1, i \neq j}^{m} [0, S_{i}] \times \prod_{i=1, i \neq j}^{m} [0, D_{i}]\), the function \( v_{j}(\cdot, s_{-j}, d_{-j}) : [0, S_{j}] \times [0, D_{j}] \rightarrow \mathbb{R} \) is strictly concave and twice differentiable.

2. The Jacobian matrix \( \bar{J} \) is strictly (row) diagonally dominant on \( \prod_{i=1}^{m} [0, S_{i}] \times \prod_{i=1}^{m} [0, D_{i}] \).

Equivalently to (1), we define the optimization mapping \((s^{*}_{j}(s_{-j}, d_{-j}), d^{*}_{j}(s_{-j}, d_{-j})) : \prod_{i=1}^{m} [0, S_{i}] \times \prod_{i \neq j}^{m} [0, D_{i}] \rightarrow [0, S_{j}] \times [0, D_{j}]\) for \( j = 1, \ldots, m \), as

\[
(s^{*}_{j}(s_{-j}, d_{-j}), d^{*}_{j}(s_{-j}, d_{-j})) = \text{arg max}_{(s, d) \in [0, S_{j}] \times [0, D_{j}]} v_{j}(s, d, s_{-j}, d_{-j}).
\]  

(10)

The goal is now to show that

**Theorem 3.3.** Under Assumption 3.2, the Nash equilibrium for the optimization problem

\[
(\mathbf{s}^{**}, \mathbf{d}^{**}) = (s^{*}_{1}(s^{**}_{-1}, d^{**}_{-1}), \ldots, s^{*}_{m}(s^{**}_{-m}, d^{**}_{-m}), d^{*}_{1}(s^{**}_{-1}, d^{**}_{-1}), \ldots, d^{*}_{m}(s^{**}_{-m}, d^{**}_{-m}))
\]  

exists and is unique.

**Proof.** Define \( T^{*} : \prod_{i=1}^{m} [0, S_{i}] \times \prod_{i=1}^{m} [0, D_{i}] \rightarrow \prod_{i=1}^{m} [0, S_{i}] \times \prod_{i=1}^{m} [0, D_{i}] \) as

\[
T^{*}(s, d) = (s^{*}_{1}(s_{-1}, d_{-1}), \ldots, s^{*}_{m}(s_{-m}, d_{-m}), d^{*}_{1}(s_{-1}, d_{-1}), \ldots, d^{*}_{m}(s_{-m}, d_{-m}))^{T},
\]

The first step is to show that \( T^{*} \) is a contraction mapping, i.e., to show that for any two distinct points \((s^{0}, d^{0}), (s^{1}, d^{1}) \in \prod_{i=1}^{m} [0, S_{i}] \times \prod_{i=1}^{m} [0, D_{i}]\), \( \|T^{*}(s^{0}, d^{0}) - T^{*}(s^{1}, d^{1})\|_{\infty} < \|(s^{0}, d^{0}) - (s^{1}, d^{1})\|_{\infty} \).

Fix \( j \in \{1, 2, \ldots, m\} \), and note that it is sufficient to show that

\[
\left| s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) - s^{*}_{j}(s^{1}_{-j}, d^{1}_{-j}) \right| < \|(s^{0}, d^{0}) - (s^{1}, d^{1})\|_{\infty},
\]

(12)

and there is nothing to prove. Thus assume that \( s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) \neq s^{*}_{j}(s^{1}_{-j}, d^{1}_{-j}) \).

Consider (12) first. If \( s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) = s^{*}_{j}(s^{1}_{-j}, d^{1}_{-j}) \), then

\[
0 = \left| s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) - s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) \right| < \|(s^{1}, d^{1}) - (s^{1}, d^{1})\|_{\infty}.
\]

Recall that \((s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}), d^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}))\), \( i = 0, 1 \) are the maximizers of (10). Then consider the following cases, based on whether the partial derivatives are zero or not.

Let

\[
x^{i}_{j} = (s^{i}_{j}(s^{0}_{-j}, d^{0}_{-j}), d^{i}_{j}(s^{0}_{-j}, d^{0}_{-j}), s^{0}_{-j}, d^{0}_{-j}), \quad i = 0, 1,
\]

and consider the case when \( \frac{\partial v_{j}(x^{0}_{j})}{\partial s_{j}} = \frac{\partial v_{j}(x^{0}_{j})}{\partial s_{j}} \). Define

\[
\phi(\theta) = \frac{\partial v_{j} \left( x^{0}_{j} + \theta(x^{1}_{j} - x^{0}_{j}) \right)}{\partial s_{j}}, \quad \theta \in [0, 1].
\]

(14)

Then, \( \phi(0) = \phi(1) \). By Rolle’s theorem, there is a \( \theta_{0} \in (0, 1) \) such that \( \phi'(\theta_{0}) = 0 \). Let \( z_{0} = x^{0}_{j} + \theta_{0}(x^{1}_{j} - x^{0}_{j}) \), then

\[
\phi'(\theta_{0}) = \sum_{i=1}^{m} \frac{\partial^{2} v_{j}(z_{0})}{\partial s_{j} \partial s_{i}} (s^{1}_{i} - s^{0}_{i}) + \sum_{i=1}^{m} \frac{\partial^{2} v_{j}(z_{0})}{\partial s_{j} \partial d_{i}} (d^{1}_{i} - d^{0}_{i}) + \frac{\partial^{2} v_{j}(z_{0})}{\partial s_{j}^{2}} (s^{*}_{j}(s^{0}_{-j}, d^{0}_{-j}) - s^{*}_{j}(s^{1}_{-j}, d^{1}_{-j})).
\]
Then, from the to triangular inequality it follows that
\[
\left| \frac{\partial^2 v_j(z_0)}{\partial s_j^2} \right| \left| s_j^*(s_{-j,0}, d_{-j}) - s_j^*(s_{-j,0}, d_{-j}) \right| \leq \sum_{i=1,i\neq j}^m \left| \frac{\partial^2 v_j(z_0)}{\partial s_j \partial s_i} \right| \left| s_i^1 - s_i^0 \right| + \sum_{i=1}^m \left| \frac{\partial^2 v_j(z_0)}{\partial s_j \partial d_i} \right| \left| d_i^1 - d_i^0 \right|.  \tag{15}
\]

From strictly diagonal dominance condition of \( \bar{J} \), it follows that
\[
\left( \sum_{i=1,i\neq j}^m \left| \frac{\partial^2 v_j(z_0)}{\partial s_j \partial s_i} \right| + \sum_{i=1}^m \left| \frac{\partial^2 v_j(z_0)}{\partial s_j \partial d_i} \right| \right) \left\| (s^0, d^0) - (s^1, d^1) \right\|_\infty \leq \left\| (s^0, d^0) - (s^1, d^1) \right\|_\infty. \tag{16}
\]

Combining inequality \([15]\) and \([16]\), we get
\[
\left| s_j^*(s_{-j,0}, d_{-j}) - s_j^*(s_{-j,0}, d_{-j}) \right| \leq \left\| (s^0, d^0) - (s^1, d^1) \right\|_\infty. \tag{17}
\]

Next, consider the case that \( \frac{\partial v_j(x_j^0)}{\partial s_j} < \frac{\partial v_j(x_j^1)}{\partial s_j} \). In this case, we assert that \( s_j^*(s_{-j,0}, d_{-j}) \leq s_j^*(s_{-j,1}, d_{-j}) \). The condition \( \frac{\partial v_j(x_j^0)}{\partial s_j} < \frac{\partial v_j(x_j^1)}{\partial s_j} \) excludes the case \( \frac{\partial v_j(x_j^0)}{\partial s_j} = \frac{\partial v_j(x_j^1)}{\partial s_j} = 0 \), which is the case that both points \( (s_j^*(s_{-j,0}, d_{-j}), d_j^*(s_{-j,0}, d_{-j})), (s_j^*(s_{-j,1}, d_{-j}), d_j^*(s_{-j,1}, d_{-j})) \) can be interior points of \([0, S_j] \times [0, D_j] \). Therefore we need to consider two scenarios:

First, when \( \frac{\partial v_j(s_{-j,0}, d_{-j})}{\partial s_j} < 0 \), corresponds to the scenario when \( s_j^*(s_{-j,0}, d_{-j}) = 0 \). Therefore, we readily get that \( 0 = s_j^*(s_{-j,0}, d_{-j}) \leq s_j^*(s_{-j,1}, d_{-j}) \).

Otherwise, we must have that \( \frac{\partial v_j(s_{-j,0}, d_{-j})}{\partial s_j} > 0 \). In this scenario, \( s_j^*(s_{-j,1}, d_{-j}) = S_j \), and thus \( s_j^*(s_{-j,0}, d_{-j}) \leq s_j^*(s_{-j,1}, d_{-j}) \).

Again, we may assume that \( s_j^*(s_{-j,0}, d_{-j}) \leq s_j^*(s_{-j,1}, d_{-j}) \), and define \( \phi(\theta) \) as in \([14]\). According to Lagrange Mean Value theorem, there is \( \theta_0 \in (0, 1) \) such that for \( z_0 = x_j^0 + \theta_0(x_j^1 - x_j^0) \), we get
\[
\phi(\theta_0) = \sum_{i=1,i\neq j}^m \frac{\partial^2 v_j(z_0)}{\partial s_j \partial s_i} (s_i^1 - s_i^0) + \sum_{i=1}^m \frac{\partial^2 v_j(z_0)}{\partial s_j \partial d_i} (d_i^1 - d_i^0) + \frac{\partial^2 v_j(z_0)}{\partial s_j^2} (s_j^*(s_{-j,1}, d_{-j}) - s_j^*(s_{-j,0}, d_{-j})) > 0.
\]

We then have
\[
- \frac{\partial^2 v_j(z_0)}{\partial s_j^2} (s_j^*(s_{-j,1}, d_{-j}) - s_j^*(s_{-j,0}, d_{-j})) < \sum_{i=1,i\neq j}^m \frac{\partial^2 v_j(z_0)}{\partial s_j \partial s_i} (s_i^1 - s_i^0) + \sum_{i=1}^m \frac{\partial^2 v_j(z_0)}{\partial s_j \partial d_i} (d_i^1 - d_i^0).  \tag{18}
\]

By the concavity of \( v_j \), the strictly diagonal dominance condition of \( \bar{J} \), and the fact that \( s_j^*(s_{-j,1}, d_{-j}) > s_j^*(s_{-j,0}, d_{-j}) \), the left hand side of \([18]\) is positive. Then, from strictly diagonal dominant property and triangular inequality, we again get \([17]\).

Similar argument shows \([13]\), and therefore since \([12]\) and \([13]\) are true for any \( j \in 1, 2, \ldots, m \), it follows that \( T^* \) is subcontracting mapping.

In particular, \( T^* \) is a continuous function, and thus by Brouwer’s fixed point Theorem there exists a fixed point for \( T^* \) in \( \prod_{i=1}^m [0, S_i] \times \prod_{i=1}^m [0, D_i] \). Such a fixed point \([11]\) is therefore a Nash equilibrium, the uniqueness of which follows from the subcontracting property of \( T^* \).

\( \square \)
4 Optimal Strategy to Recover Shortfall

We now turn to discuss one more case when the banks in the system face a shortfall and need to raise funds to cover it. This is a more classical scenario used e.g. in Bichuch and Feinstein (2019). Without loss of generality, assume that all banks \( j = 1, ..., m \) in the system face shortfall \( M_j > 0 \), otherwise, the system can be shrunk accordingly, and that all the banks are solvent. In other words they can sell and borrow so as to be able to cover their debt. Otherwise, similar to Bichuch and Feinstein (2019), we may assume that these banks will not make any transaction and will be taken over by the regulator. We also continue with our previous assumptions that the bank \( j \), \( j = 1, 2, 3, ..., m \) can sell at most \( S_j \) shares of its stock and can borrow at most \( D_j \) dollars, and that banks can raise fund through stock sale and borrowing. Then, the cost of such a transaction as a function of \((s, d)\) to the bank \( j \) is given by

\[
w_j(s_j, d_j, s_{-j}, d_{-j}) = s_j(p_j(0, 0; 0, 0) - p_j(s_j, d_j, s_{-j}, d_{-j})) + r_j(s_j, d_j, s_{-j}, d_{-j})d_j.
\]

Here, the first and the second terms represent the opportunity loss due to selling the stock at the reduced stock price, and the cost of the newly issued debt respectively. The purpose of each bank is to recover its shortfall with financing cost as small as possible. That is for the bank \( j \) minimizes

\[
\min_{(s_j, d_j)\in[0, S_j] \times [0, D_j]} w_j(s_j, d_j; s_{-j}, d_{-j}), \text{ subject to } d_j + s_j p_j(s_j, d_j, s_{-j}, d_{-j}) = M_j. \tag{19}
\]

As we argued in previous section, debt financing does not pull down the stock price significantly, especially when the amount of borrowing is not so big. As the shortfall needed to be recovered, \( M_j \), is not very large, for any given number of shares of its stock \( s_j \), the total amount of raising fund increases with respect to borrowing \( d_j \). These facts lead to the assumptions below, with which we can prove the existence and uniqueness of Nash equilibrium strategy in this scenario.

**Assumption 4.1.**

1. \((s, d)\mapsto d_j + s_j p_j(s_j, d_j, s_{-j}, d_{-j})\) is a concave, twice continuously differentiable function, \( j = 1, 2, ..., m \).

2. \( D_j > M_j, j = 1, 2, ..., m \).

3. For each \( j = 1, ..., m \), given \((s_{-j}, d_{-j})\) \(\in\prod_{i=1, i\neq j}^{m}[0, S_i] \times \prod_{i=1, i\neq j}^{m}[0, D_i] \), \( w_j(\cdot, \cdot; s_{-j}, d_{-j}) : [0, S_j] \times [0, D_j] \mapsto \mathbb{R} \) is a strictly convex twice differentiable function.

4. Let \( \bar{F}(s, d) = \left( \frac{\partial w_1}{\partial s_1}, \frac{\partial w_1}{\partial d_1}, \cdots, \frac{\partial w_m}{\partial s_m}, \frac{\partial w_m}{\partial d_m} \right)^T \), and let \( \bar{J}(s, d) \) be the Jacobian matrix of \( \bar{F} \). 

\( \bar{J}(s, d) \) is a strictly diagonal dominant matrix with any \((s, d)\) \(\in\prod_{i=1}^{m}[0, S_i] \times \prod_{i=1}^{m}[0, D_i] \).

With these assumptions, we can prove the existence and uniqueness of Nash equilibrium.

**Theorem 4.2.** Under Assumptions [4.1], there exists an unique Nash equilibrium for the minimization problem [19], selling stock and borrowing strategy:

\[
(s^*, d^*) = (s_1^*(s_{-1}^*, d_{-1}^*), ..., s_m^*(s_{-m}^*, d_{-m}^*)) \in \prod_{i=1}^{m}[0, S_i] \times \prod_{i=1}^{m}[0, D_i].
\]

**Proof.** First, let’s relax the restriction and show that there is an unique Nash equilibrium for [20]

\[
\min_{(s_j, d_j)\in[0, S_j] \times [0, D_j]} w_j(s_j, d_j; s_{-j}, d_{-j}), \text{ subject to } d_j + s_j p_j(s_j, d_j, s_{-j}, d_{-j}) \geq M_j. \tag{20}
\]
For any \( j = 1, 2, ..., m \), let \( U_j \subset \prod_{i=1}^{m} [0, S_i] \times \prod_{i=1}^{m} [0, D_i] \) be a set that satisfies constraint \( d_j + s_j p_j(s_j, d_j, s_{-j}, d_{-j}) \geq M_j \) and \( U = \bigcap_{i=1}^{m} U_j \). According to Assumption 4.1.2, \( (0, D_1, D_2, ..., D_m) \) must be in \( U \), then we know \( U \) is not empty. By concavity and continuity of function \( d_j + s_j p_j(s_j, d_j, s_{-j}, d_{-j}) \), Assumption 4.1.2, \( U_j \) is compact and convex for each \( j = 1, 2, ..., m \), therefore so is \( U \).

Let also \( U_j^{(s_{-j}, d_{-j})} = \{(s_j, d_j) : (s, d) \in U_j\} \), \( j = 1, ..., m \). Then, any Nash equilibrium for the constrained optimization problem \( 20 \), must also be a Nash equilibrium to the optimization problem

\[
\min_{(s_j, d_j) \in U_j^{(s_{-j}, d_{-j})}} w_j(s_j, d_j; s_{-j}, d_{-j})
\]

as well. And the opposite is also true. Hence, we only need to show the existence and uniqueness of Nash equilibrium of optimization problem \( 21 \) and this can be proved by similar argument of Theorem 3.3 with Assumption 4.1.4 and 4.1.3.

Next, let’s prove that it is also the Nash equilibrium of constrained optimization problem \( 19 \). Assume \( (s^{**}, d^{**}) = (s^*_1(s^{**}_1, d^{**}_1), ..., s^*_m(s^{**}_m, d^{**}_m), d^*_1(s^{**}_1, d^{**}_1), ..., d^*_m(s^{**}_m, d^{**}_m)) \) is the Nash equilibrium of optimization problem \( 20 \), with some \( j \in \{1, 2, ..., m\} \) such that \( d^*_j(s^{**}_j, d^{**}_j) + s^*_j(s^{**}_j, d^{**}_j)p_j(s^*_j(s^{**}_j, d^{**}_j), d^*_j(s^{**}_j, d^{**}_j), s^*_j, d^*_j) > M_j \). Then, \( (s^*_j(s^{**}_j, d^{**}_j), d^*_j(s^{**}_j, d^{**}_j)) \) must be a interior point of \( U_j^{(s^{**}_j, d^{**}_j)} \). Hence, we can reduce \( d^*_j(s^{**}_j, d^{**}_j) \) a little to \( d^*_j(s^{**}_j, d^{**}_j) \) to make sure that \( (s^*_j(s^{**}_j, d^{**}_j), d^*_j(s^{**}_j, d^{**}_j)) \) is still in \( U_j^{(s^{**}_j, d^{**}_j)} \) so that the shortfall coverage constraint is still satisfied. With the fact that given \( s_j, s_{-j}, d_{-j}, w_j \), \( w_j \) is strictly decreasing function with respect to \( d_j \), we have \( w_j(s^*_j(s^{**}_j, d^{**}_j), d^*_j(s^{**}_j, d^{**}_j), s^*_j, d^*_j) < w_j(s^*_j(s^{**}_j, d^{**}_j), d^*_j(s^{**}_j, d^{**}_j), s^*_j, d^*_j) \), which is a contradiction with the definition of Nash equilibrium. That is the Nash equilibrium \( (s^{**}, d^{**}) = (s^*_1(s^{**}_1, d^{**}_1), ..., s^*_m(s^{**}_m, d^{**}_m), d^*_1(s^{**}_1, d^{**}_1), ..., d^*_m(s^{**}_m, d^{**}_m)) \) of optimization problem \( 20 \) must satisfy the equality constraint for optimization problem \( 19 \) of any \( j = 1, 2, ..., m \). Therefore, there exist an unique Nash equilibrium of optimization problem \( 19 \). □

5 Empirical Analysis

We now investigate if the current financial system is more stable than it was before the crisis by considering the ability of banks to raise extra capital first through selling stock alone, and then through both selling stock and borrowing. We concentrate on two examples: JP Morgan Chase (JPM) and Citi bank. Both banks were strong before the financial crisis 2008. The difference between them is that the former became even (relatively) stronger by absorbing Bear-Stearns, while the latter was on the brink of failing and survived, arguably, only due to government’s help.

In this section, we use the historical daily market capitalization and quarterly equity book values obtained from Bloomberg to estimate the daily book-to-price ratio for both JPM and Citi bank. We also use Bloomberg to obtain the daily overnight interest rate for the two banks, and the market Overnight Interest Swap (OIS) rate which is treated as benchmark of the whole market. Finally, we use quarterly interest expense data for each bank and the quarterly overnight interest rate to estimate its short-term loan \( L_j, j \in \{JPM, C\} \). All the data is between 03/31/1998 to 06/23/2017.

5.1 Estimate Overnight Interest Rate Function

Denote \( y_j(s, d) \) to be the book-to-price ratio of bank \( j \in \{1, ..., m\} \). Recall that we previously assumed that the short-term rate is a function of the book-to-price ratios. Therefore, we write
\( r_j(s_j, d_j, s_{-j}, d_{-j}) = \tilde{r}_j(y_1(s, d), ..., y_m(s, d)) \). We then proceed to approximate the overnight interest rate functions \( \tilde{r}_j \) of bank \( j \) by using Ordinary Least Square method, as follows. The general regression model for the overnight interest rate \( \tilde{r}_j \) is

\[
\tilde{r}_j(y_1(s, d), ..., y_m(s, d)) \sim \beta_0 + \beta_1 y_j + \beta_2 \sum_{i \neq j} y_i + \beta_3 OIS,
\]

where \( OIS \) is the overnight interest swap rate. The parameters of linear estimations for JPM and Citi bank are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>JPM</th>
<th>Citi</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_0 )</td>
<td>-0.044847</td>
<td>-0.050231</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.10478</td>
<td>0.12185</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.11694</td>
<td>0.13361</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>1.0321</td>
<td>1.0265</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.972</td>
<td>0.969</td>
</tr>
</tbody>
</table>

The high \( R^2 \)s suggest that the linear models are very good approximations for the short-rates \( \tilde{r}_j, j = \{1, 2, \ldots, m\} \).

### 5.2 The Optimal Strategies to Raise Cash

We first estimate the optimal strategy to raise money by selling stock alone. We use linear price impact function from Example 2.3 and choose \( a_{JPM} = \frac{1}{900} \) for JPM and \( a_C = \frac{1}{150} \) for Citi bank, whose denominators are around one third of the number of outstanding shares in millions for each bank. We use \( S_{JPM} = 400 \) for JPM and \( S_C = 200 \) for Citi, to satisfy Assumption 2.1. We choose \( \epsilon = 10^{-5} \) to be much smaller than \( a_{JPM} \wedge a_C \). The left and the right graphs of Figure 1 show the maximal amount of funds that JPM and Citi can raise respectively through only selling its stock as a function of time. As expected, there is a peak of market’s confidence, before the financial crash, for both banks. Whereas during the height of crash the amounts decrease dramatically. For JPM the minimum is reached some time after the crash and the acquisition of Bear-Stearns, as the markets stabilized, and has been (almost) monotonically increasing since, as both the market’s confidence in JPM and banking system overall recovers. Unlike JPM, the recovery of Citi bank has been less significant. The main reason for these different recoveries is shown in Figure 2. Although there is a recovery of market confidence, for which we use price-to-book ratio as a proxy, for Citi bank, it has not recovered as much as it had for JPM. The continuous low market confidence prevents Citi bank from raising funds through stock sale.
Next, we repeat the optimization when borrowing is also allowed. We use the linear inverse demand price function of Example 3.1. For the empirical experiment in this part, we continue to use \( a_{JPM} = \frac{1}{900}, S_{JPM} = 400 \) for JPM and \( a_C = \frac{1}{450}, S_C = 200 \) for Citi to be consistent with previous setting, which ensures that the price function \( p_j(s_j, d_j, s_{-j}, d_{-j}), j \in \{JPM, C\} \) satisfy Assumption 3.2. Moreover, we choose \( b_j = 1500, D_j = 15, j \in \{JPM, C\} \) for both banks which is similar to the size of market capitalizations in billions of the banks during 2008 financial crisis and to match the fact that a little borrowing has little effect on the stock price. Finally, we choose \( \epsilon_1 = 10^{-5}, \epsilon_2 = 10^{-6} \) to make sure that the effect of other banks are not as significant as that of the bank itself. Figures 3 and 4 are plotted based on the above parameters.

Figure 3 provides the maximal cash each bank can raise. Similar to the scenario in which borrowing is not an option, we observe that both banks experienced a big decline in the amount of money they can raise during the crash, and both recovered from these lows. However, Citi’s recovery is only partial as opposed to a more robust recovery by JPM. Figure 4 shows the maximum amount of funds that can be raised by JPM and Citi through both stock sale and borrowing component-wise. Recall both stock sale and borrowing, incurs costs and reduces market confidence. Therefore,
the actual amounts that can be raised in Figure 3 are the sum of the red and blue curves minus the black dashed curve in Figure 4.

Additionally, comparing Figure 3 with Figure 2, both of the banks can raise more funds when borrowing is allowed, especially Citi. It appears that JPM cannot raise substantially more funds than before by using stock sale and borrowing together. The main reason for this is shown in the left graph of Figure 4, as the cost significantly increases after the financial crisis, which forces JPM to rely more stock sale than on borrowing. Moreover, Figure 4 points out that before the crash, the optimal strategy for Citi to raise funds was to sell its stock with little borrowing, while after the crash it became optimal to rely almost exclusively on borrowing. This is not surprising, since the government ultimately converted its loan to stock, and signifies a continuation of low confidence in the bank, as debt holders are reimbursed before the shareholders, who are always the last to be reimbursed in case of a default. Additionally it is commonly assumed that Citi’s too big to fail status will prevent any losses to the debt holders. The situation is almost reversed for JPM, whose balance sheet now implies that it can raise significant funds by selling stock, due to high market confidence. This in addition to borrowing funds cheaply, likely partially due to its too big to fail status. These findings are also supported in Figure 2 illustrating the price-to-book ratio for both banks, which we use as our proxy to market confidence. In Figure 2 the lowest point of price-to-book ratio is above 0.6 for JPM (which is around 1 for most of the time), while the average price-to-book ratio for Citi after the crash is only around 0.5. The relatively more robust market confidence of JPM makes it possible for the bank to raise funds through stock sales even in financial crisis period. Moreover, comparing with stock sale, borrowing is less sensitive to market confidence, although it still declined during the crash.

Figure 3: Total Cash can be Raised
5.3 Assumption Verification

For linear price impact function, the choice $a_j < \frac{1}{2S_j}$, ensures that Assumption 2.1 always holds. We now verify that Assumption 3.2 holds using the parameter choice of Section 5.2 for the two banks case. It is sufficient to check that

\begin{align}
-\frac{\partial^2 v_j}{\partial s_j^2} &> \left| \frac{\partial^2 v_j}{\partial d_i \partial s_j} \right| + \left| \frac{\partial^2 v_j}{\partial s_i \partial s_j} \right| ,
\end{align}

(22)

\begin{align}
-\frac{\partial^2 v_j}{\partial d_j^2} &> \left| \frac{\partial^2 v_j}{\partial s_i \partial d_j} \right| + \left| \frac{\partial^2 v_j}{\partial s_j \partial d_j} \right| + \left| \frac{\partial^2 v_j}{\partial d_i \partial d_j} \right| .
\end{align}

(23)

In practice in order to raise funds a bank will not sell a handful of stocks, but will have to sell a substantial amount. Therefore, we assume that for $j \in \{JPM, C\}$ there exists $\eta_j > 0$ such that bank $j$ will not sell less than $\eta_j$ shares, and we restrict the domain of $s_j$ to $[\eta_j, S_j]$. It is not hard to verify that our results still hold with this restriction. Using our findings in Table 1 and the fact that $\frac{s_j}{S_j} < y$, for any $(s, d) \in \prod_{j=1}^2 [\eta_j, S_j] \times [0, D_j]$, it can be checked that all the second derivatives $\frac{\partial^2 v_j}{\partial s_j^2}, \frac{\partial^2 v_j}{\partial d_j^2}, \frac{\partial^2 v_j}{\partial s_i \partial s_j}, \frac{\partial^2 v_j}{\partial s_i \partial d_j}, \frac{\partial^2 v_j}{\partial s_j \partial d_j}, \frac{\partial^2 v_j}{\partial d_i \partial d_j} < 0$ are strictly negative. Moreover, since the numerical value of $L_C, L_{JPM}$ are around $10^6$, when the parameters $a_j, b_j, j \in \{C, JPM\}, \epsilon_1, \epsilon_2$ are small, then $\eta_j \geq D_j$ and $a_j > b_j D_j$ are sufficient condition for (22) and (23) to hold. Thus, the functions $v_j$, $j \in \{C, JPM\}$ given in (9) that is being used for the empirical calculation satisfies all the assumption needed in Section 3.

6 Conclusion

In this paper, we consider the maximum amount of funds that banks can raise either through stock sales alone, or through both stock sales and borrowing, and the problem if raising funds to cover given shortfall. We have created a simple model, incorporating price-to-book ratio as market’s confidence proxy in this optimization problem. We have shown the existence and uniqueness of Nash equilibrium in both maximization problems, and performed a time series empirical analysis of two banks to show how they weathered through the last financial crisis.
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