Printable Topics in Mathematics

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A Collection of p-Norm Balls

Notions of Distance Ranging from Euclidean to Manhattan Taxicabs and Beyond

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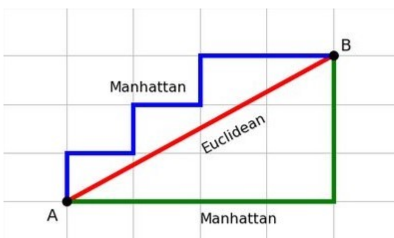
Classic Distances: Euclidean & Manhattan

The notion of distance we all learned in school is the Euclidean distance. Followings from the Pythagorean theorem, we compute the *Euclidean norm* of a point (x, y) as

$$\|(x,y)\|_{Euclidean} = \sqrt{x^2+y^2}\ .$$

When driving around a city, this is not the right notion of distance as we only drive on streets. Instead, we often want to use the so-called *Manhattan norm* of (x, y) adding together the north-south travel with east-west travel

$$\|(x,y)\|_{Manhattan} = |x| + |y|\ .$$



These two correspond to $p = 1, 2$ norms respectively.

3D Model Viewer and Links

- `.stl` files and 3D model viewer at `printables.com/model/244001`
- This `.pdf` is available at `ams.jhu.edu/~grimmer/pNorm.pdf`

DISTANCES are often foundational to describing geometry and play a fundamental role throughout calculus and analysis. Formally a *norm* is a function that takes an input v and returns a number representing its size $\|v\|$. To be a norm, it needs to satisfy several conditions (namely, the Triangle Inequality, Positive Definiteness, and Absolute Homogeneity).

Here we depict one particular family of norms generalizing the two common notions of distance to the left. We denote the p-norm of a vector v in dimension n for any $p \geq 1$ as

$$\|v\|_p = \left(\sum |v_i|^p\right)^{1/p}\ .$$

Taking the limit as $p \to \infty$, you can verify $\|v\|_\infty = \max\{|v_i|\}$.

Dual Norms and Mapping Between Corners and Faces

Let $w^\top v$ denote the inner product of two vectors. Given a norm $\|v\|$ for vectors v, we can construct its *dual norm* $\|w\|_*$ as

$$\|w\|_* = \max_{\|v\|=1} w^\top v\ .$$

Each p-norm's dual norm is the q-norm where $1/p + 1/q = 1$. Inspecting the pairs of dual unit balls below, we see the extreme corner points of a ball align with flat regions in the dual ball and vice versa. For example, $p = 1$ below has 6 corners and 8 faces while its dual value $p = \infty$ has 8 corners and 6 faces.

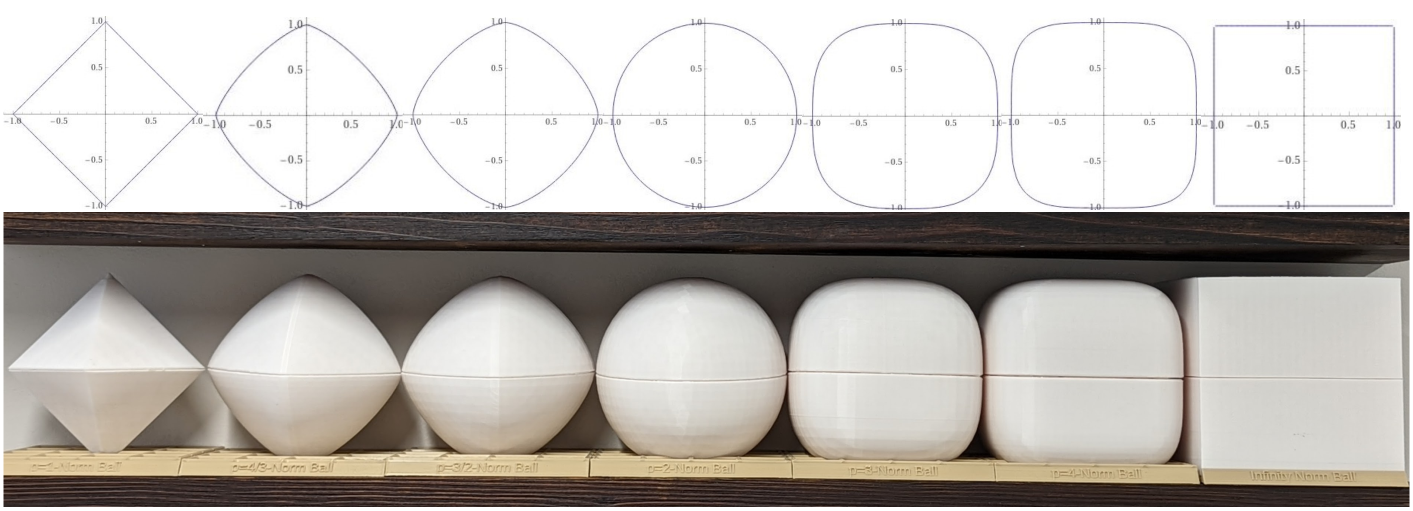


Figure 1. The first and second rows show the unit balls $\{v \mid \|v\|_p \leq 1\}$ in two dimensions and three dimensions, respectively. From left to right, the norms correspond to $p = 1, 4/3, 3/2, 2, 3, 4, \infty$, making the outermost norms dual, second outermost norms dual, etc. Each 3D p-norm ball has the $z = 0$ plane marked, cutting through its middle. This cross section is exactly the 2D p-norm ball.