Printable Topics in Mathematics

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Schatten Matrix p-Norm Balls

A Generalization of p-Norms to Matrices and a View of Low Rank Matrices

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Classic Schatten Norms $p \in \{1, 2, \infty\}$

The *Frobenius Norm* of a matrix A is the sum of its entries squared: $\|A\|_F^2 = \sum A_{ij}^2$. This aligns with the classic Euclidean norm for vectors. Consequently, the Schatten 2-norm below is a sphere. In terms of singular values, this is exactly their sum squared. Hence $\|A\|_F = \|A\|_2$.

The *Operator Norm* of a matrix A measures how large Av can be: $\|A\|_{op} = \sup_{\|v\|_2=1} \|Av\|_2$, utilizing the vector two-norm. In terms of singular values of A, the largest one captures this largest distortion. Hence $\|A\|_{op} = \|A\|_\infty$.

The *Nuclear Norm* adds up the singular values of A directly: $\|A\|_{nuclear} = \|A\|_1$. This norm has found widespread usage in optimization over low-rank matrices.

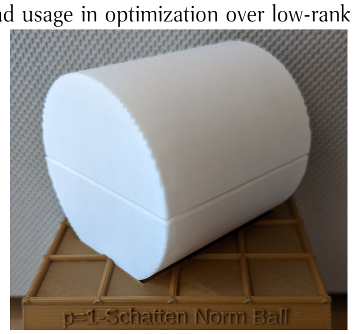

Figure 1. The low-rank matrices can be seen in the Schatten 1-norm ball in the two rings of the extreme points, bounding the barrel-like shape. The far ring is all the rank one, positive semidefinite matrices $xx^\top$ whereas the near ring is all rank one, negative semidefinite matrices $-xx^\top$.

3D Model Viewer and Links

- ❑ `.stl` files and 3D model viewer at `printables.com/model/244711`
- ❑ This `.pdf` is available at `ams.jhu.edu/~grimmer/Schatten.pdf`

Norms are critical to calculus quantifying small changes and linear algebra measuring sizes of matrices/operators. Previously (`ams.jhu.edu/~grimmer/pNorm.pdf`), we considered the p-norm of a vector v for $p \geq 1$ defined as

$$\|v\|_p = \left(\sum |v_i|^p\right)^{1/p} .$$

Here we illustrate and print the natural extension of this to matrix norms: The *Schatten p-norm* of a matrix A for $p \geq 1$ is

$$\|A\|_p = \left(\sum \sigma_i(A)^p\right)^{1/p} .$$

where $\sigma_i(A)$ is the ith smallest singular value of A. Mirroring the vector p-norm, we say $\|A\|_\infty = \max\{\sigma_i(A)\}$.

Dual Norms and Mapping Between Corners and Faces

The dual matrix norm of a given matrix norm $\|\cdot\|$ is defined as $\|W\|_* = \max_{\|V\|=1} \mathrm{trace}(WV)$, using the trace inner product. Like vector norms, the Schatten p-norm is dual to the Schatten q-norm with $1/p + 1/q = 1$ (A good exercise: Why?). Four pairs of dual norms $(1, \infty), (4/3, 4), (3/2, 3), (2, 2)$ are below. For example, the barrel-shaped 1-norm ball is dual to the top-shaped ∞-norm ball (their points and faces map to each other).

Embedding Matrices in 3D for Printing

In general, it is tricky to visualize a space of matrices. To do so in 3D, we need matrices with exactly three degrees of freedom. Symmetric 2×2 matrices give us exactly this, parameterized by (x, y, z) below (rescaling y to balance it appearing twice)

$$(x, y, z) \mapsto \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} .$$

Figure 2. From left to right, Schatten p-norm balls of symmetric 2×2 matrices with $p = 1, 4/3, 3/2, 2, 3, 4, \infty$. Each ball has the $y = 0$ plane marked, cutting through its middle. This cross section is exactly the 2D vector p-norm ball (A good exercise: Why?).