Printable Topics in Mathematics

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Max-Cut and its Approximations

Applications in Graph Theory, Statistical Physics, and Spectrahedral Geometry

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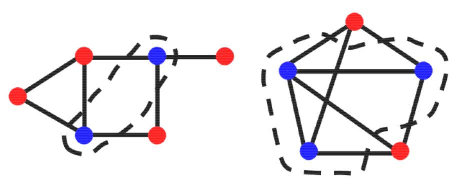
A Motivating Example: Teambulding

Suppose you are organizing a teambuilding exercise among your company/class/dorm/etc. You need to split everyone into two groups with the goal of separating existing friends, so new connections can form. You can represent the existing friendships as a graph, each node (i.e., vertex) represents a person and an edge is between two people if they are friends. Then the proposed goal is to separate them into two groups, cutting up as many edges as possible.

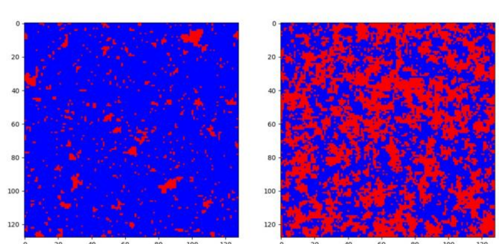
Below are two example social networks with the maximum cut shown, dividing it into two groups, red and blue.



Although these can be computed manually for small graphs, Karp showed this task is NP-Complete in general. So there is little hope for designing scalable, exact algorithms. One famous approximation (based on the spectral approximation ideas illustrated herein) of Goemans and Williamson always solves the problem within 12.215% of optimal.

Applications in Statistical Physics

Another problem of splitting up a complex network arises when modeling crystal/metal lattices. Each atom possesses a spin/magnetic orientation that interacts (aligning or repulsing from its nearby neighbors). The ***Ising model*** describes a classic version of this. Computing the lowest energy state or configuration (and hence most stable, in some entropic sense), corresponds to finding a partition of the lattice into two groups (up and down alignment). For example, below are two 2D lattices simulated with regions of up and down magnetic orientation colored red and blue.



MAXIMALLY CUTTING a graph into two halves serves both as a useful application model (see left column) and as a fundamental task in graph theory and optimization.

Formally, given n elements to cut up, denote a division into two groups by a vector $x \in \{\pm 1\}^n$ assigning $x_i = 1$ if the ith element is in the first group and $x_i = -1$ if it is in the second group. Suppose placing elements i and j together has reward A_{ij} (potentially negative) and separating them has reward $-A_{ij}$. Then maximizing the total reward over all cuts corresponds to

$$\begin{cases} \max & \sum_{ij} A_{ij} x_i x_j \qquad (= x^\top A x) \\ \text{s.t.} & x \in \{\pm 1\}^n \ . \end{cases}$$

As an aside, this problem is NP-hard (as it models the NP-Hard graph theory problem described to the left). So efficiently computing a maximizing assignment is likely impossible at scale.

Two Equivalent Problem Statements

The above quadratic optimization optimizes over the corners of a hypercube (below is an $n = 3$ hypercube, known as a cube). This quadratic problem can be rewritten as linear matrix optimization, using the trace inner product $\langle A, X\rangle = \mathrm{trace}(AX)$:

$$\begin{cases} \max & x^\top A x \\ \text{s.t.} & x \in \{\pm 1\}^n \end{cases} = \begin{cases} \max & \langle A, X\rangle \\ \text{s.t.} & X = xx^\top \\ & x \in \{\pm 1\}^n \ . \end{cases}$$

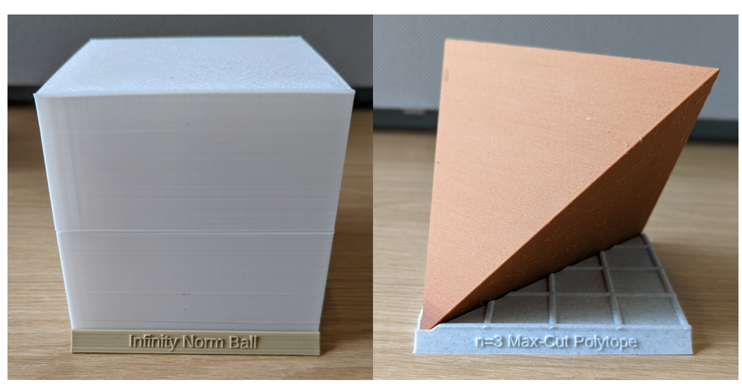


Figure 1. The convex hull of $x \in \{\pm 1\}^3$ and the points (a, b, c) with $\begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$ in the convex hull of $\{X = xx^\top \mid x \in \{\pm 1\}^3\}$.

3D Model Viewer and Source File Details

Disclaimer before you dive into these files: I am a mathematician professionally with only a self-taught/amateur background in three-dimensional printing and modeling.

- ❏ `.stl` files and 3D model viewer are available at

  `printables.com/model/239579`

- ❏ `.nb` Mathematica file is available at

  `github.com/profgrimmer/MaxCut`

- ❏ This `.pdf` is available at

  `ams.jhu.edu/~grimmer/MaxCut.pdf`

Inner and Outer Approximations

The problem of computing a max cut amounts to maximizing the linear function $\langle A, X\rangle$ over the convex hull from Figure 1:

$$MC = \texttt{convexHull}\{X = xx^\top \mid x \in \{\pm 1\}^n\}\ .$$

Note this set is a polyhedron (it is constructed from a finite collection of flat faces). Alas, this polyhedron blows up in complexity as n grows, having exponentially many corner points.

Since every matrix in MC is positive semidefinite with diagonal entries equal to one, an outer approximation is given by

$$SR = \{X \mid \texttt{diag}X = 1, X \succeq 0\} \supseteq MC\ .$$

This set is a spectrahedron, a useful generalization of polyhedrons. Consequently optimizing over SR is a semidefinite program problem which can be efficiently solved using interior point methods, avoiding the exponential blowup inherent to MC.

The following clever inner approximation of MC (studied by Nesterov and Hirschfeld) shrinks SR by an arcsin function:

$$TA = \{f(X) \mid \texttt{diag}\ X = 1, X \succeq 0\} \subseteq MC$$

where $f(X)$ entry-wise applies $x_{ij} \mapsto \frac{2}{\pi}\arcsin(x_{ij})$. Note this mapping does not change ± 1 entries, and so the target matrices $X = xx^T$ with $x \in \{\pm 1\}^n$ remain unchanged. This shrinkage TA of SR exactly recovers MC when $n = 3$ (below) and gives a strict inner approximation when $n \geq 4$ (to the right).

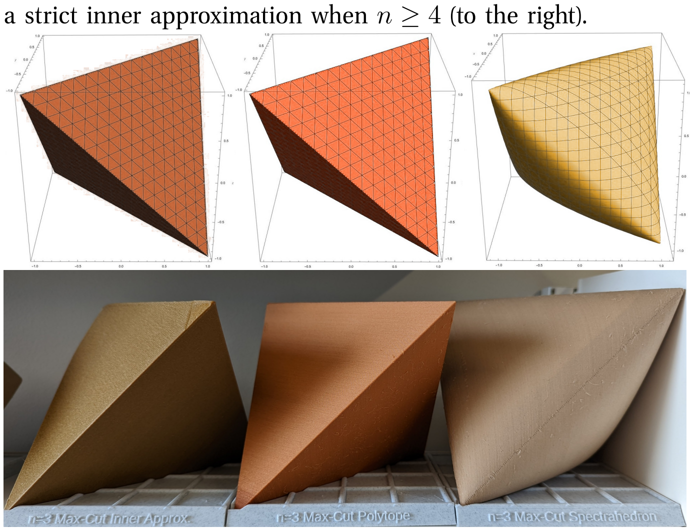


Figure 2. For $n = 3$, from left to right are the inner, true, and outer approximations $TA \subseteq MC \subseteq SR$, plotting possible values of (a, b, c), matching Figure 1's embedding.

Viewing Slices of Higher Dimensional Shapes

More interesting differences between the max-cut polytope, MC, and its inner/outer approximations, TA and SR, can be seen when $n > 3$. Lets do $n = 4$. There are six degrees of freedom in symmetric 4×4 matrices with all ones on the diagonal,

$$\left\{ \begin{bmatrix} 1 & a & b & d \\ a & 1 & c & e \\ b & c & 1 & f \\ d & e & f & 1 \end{bmatrix} \mid (a, b, c, d, e, f) \in \mathbb{R}^6 \right\}\ .$$

To plot/print these, we need to restrict down to just three degrees of freedom. Below we give three different slices of this six dimensional space, requiring three different entries be zero.

These approximations take on a greater variety of shapes and structures than those with $n = 3$ had. For example, in Figure 3, the three-dimensional slice of MC shown is a rhombic dodecahedron whose outer approximation is a sphere and in Figure 4, the inner approximation fails to be convex.

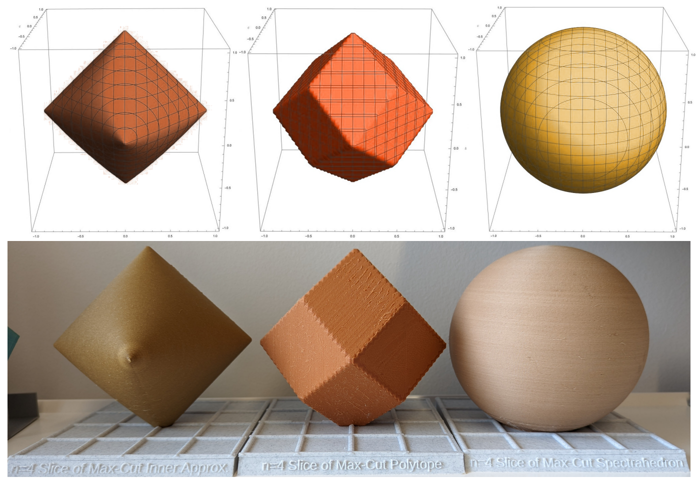


Figure 3. For $n = 4$, the sets $TA \subseteq MC \subseteq SR$ shown when restricted to $c = e = f = 0$, plotting possible values of (a, b, d).

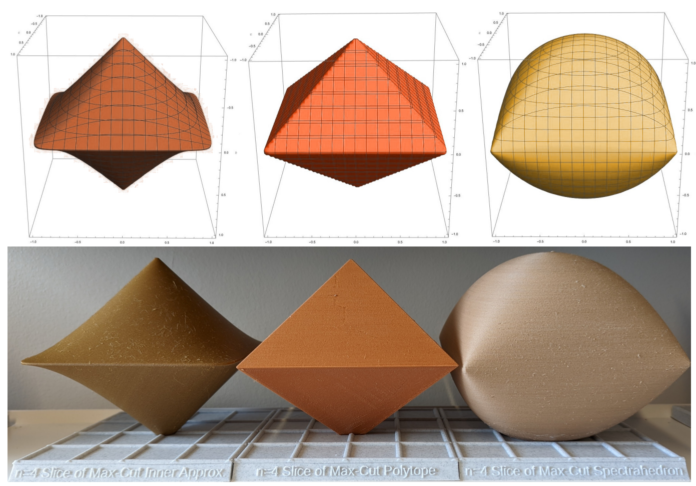


Figure 4. For $n = 4$, the sets $TA \subseteq MC \subseteq SR$ shown when restricted to $b = d = e = 0$, plotting possible values of (a, c, f).

One final slice of the 4×4 *setting.* The last interesting slice in this setting is given by restricting $TA \subseteq MC \subseteq SR$ to have $d = e = f = 0$, which gives exactly the same sets shown in Figure 2 (A good exercise: Why?). All other selections of three entries to zero out in the 4×4 matrix yield rotations of the examples in Figures 2, 3, and 4 (Another good exercise: Why?).