The Klee-Minty Cube

A Classic Case of Having Exponentially Slow Runtime for the Simplex Method

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Printable Topics in Mathematics

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Worstcase Analysis of Simplex Methods

The example (first proposed in 1972) discussed here shows the simplex method when applied to a linear program in dimension n with m constraints, can be exponentially slow in terms of n and m. However, in practice, the simplex method has a superb track record of fast convergence. Hence more theory was needed to explain this...

Randomized Analysis of Simplex Methods

In the 70s and 80s, the average case behavior of the simplex method was shown to be linear. Namely, Linear Program sampled with entirely random data (normally distributed) will typically only require a few pivots to solve.

Smoothed Analysis of Simplex Methods

In 2001, Spielman and Teng proved a much more direct result. Given any linear program, if a small amount of noise is added, then the simplex method will, on average, terminate in a polynomial time. This explains the fast practical performance: all linear programs are near easy ones.

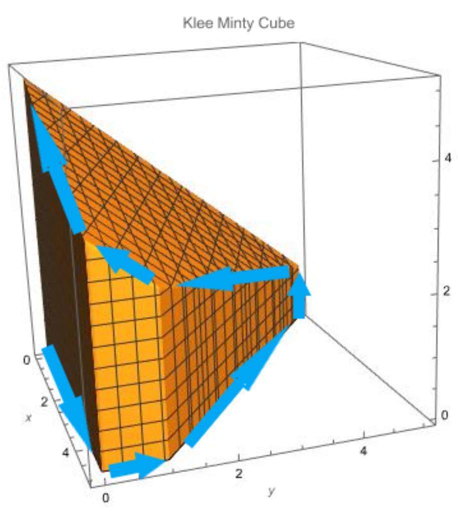

Figure 2. The simplex method visiting all 2^n corners.

3D Model Viewer and Links

- ❑ `.stl` files and 3D model viewer at `printables.com/model/390835`
- ❑ This `.pdf` is available at `ams.jhu.edu/~grimmer/Klee.pdf`

Linear Programs are a widely used optimization model with a long history of impact. These are problems of maximizing a linear objective over linear inequality constraints.

Definition of the Simplex Method

Perhaps the most classic method for solving Linear Programs is the Simplex Method, which dates back to the late 1940s. This algorithm works by moving from corner to corner of the feasible region, improving its objective. Since there are only finitely many corners, this method eventually yields an optimal corner (a full proof of this fact requires more careful reasoning). The example here shows this can visit exponentially many corners

Definition of the 3D Klee-Minty Cube

The following linear program is a classic example of Klee and Minty of the Simplex Method's potentially slow performance. Figure 1 shows the 3D-printed feasible region and Figure 2 shows a path visiting 2^n corners but always improving the objective.

$$\begin{cases} \max & 4x_1 + 10x_2 + 25x_3 \\ \text{s.t.} & x_1 \leq 5 \\ & 4x_1 + 5x_2 \leq 25 \\ & 8x_1 + 20x_2 + 25x_3 \leq 125 \\ & x \geq 0 \ . \end{cases}$$

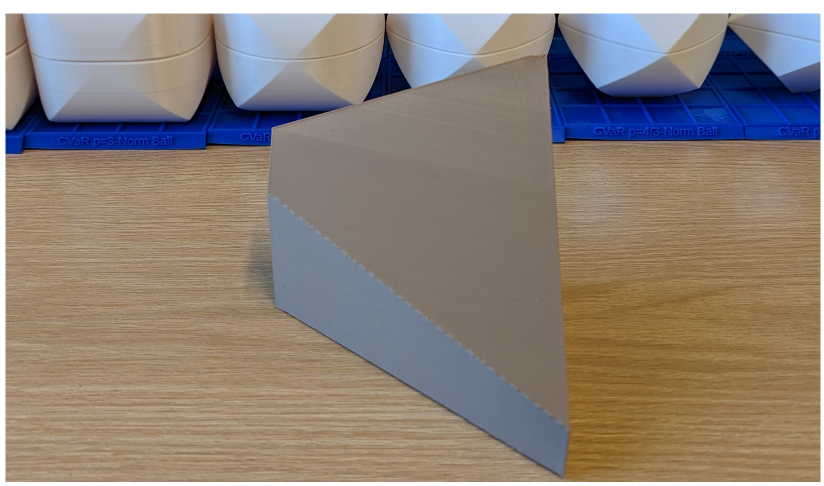

Figure 1. The Klee Minty Cube in $\mathbb{R}^n$ for the visualizable case of $n = 3$. Note in general this has only $2n$ faces but 2^n corners.