Printable Topics in Mathematics

Benjamin Grimmer

Johns Hopkins, AMS

A Collection of Induced Norm Balls

Applications to Matrix Analysis and Open Questions in Computational Complexity

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A Few Notations

Denote the *p-norm* of a vector v for $p \geq 1$ as

$$\|v\|_p = \left(\sum |v_i|^p\right)^{1/p} .$$

In another project, we printed the balls $\|v\|_p \leq 1$ (see `ams.jhu.edu/~grimmer/pNorm.pdf`):

Denote the *dual value* of $p \geq 1$ as the unique $p^* \geq 1$ solving $1/p + 1/p^* = 1$. Pairs of dual numbers play a fundamental role in understanding p-norms. The norm $\|\cdot\|_{p^*}$ is called the *dual norm* to $\|\cdot\|_p$. The seven balls above form four pairs of dual norms: $(1, \infty)$, $(4/3, 4)$, $(3/2, 3)$, $(2, 2)$.

Computing Induced Norms and Open Research Questions

The difficulty of computing $\|A\|_{p\to q}$ depends on the choice of p and q. For some, it's easy (polynomial-time computable), for some, it's hard (NP-Complete), and for some, it's an open question.

When it's easy. If $p = q = 2$, the induced norm corresponds to the largest singular value of A. If $p = 1$, then the defining supremum attains its maximum value at one of the extreme points of the ℓ_1 ball (which can be quickly enumerated). If $q = \infty$, Lemma 2 below reduces this to when $p = 1$. No other easily computed cases are known.

When it's hard. First (Rohn, 2000) showed computing the $\infty \to 1$ norm is NP-hard. Then (Steinberg, 2005) showed $p > q$, (Hendrickx et al, 2010) showed $p = q \notin \{1, 2, \infty\}$, (Bhattiprolu,2018) showed $1 < p < q < 2$ and $2 < p < q < \infty$, and (Barak et al, 2014) showed $2 \to 4$.

When it's open. When $1 < p < 2$ and $2 < q < \infty$, this question is open. Figure 2 constructs all of these norm balls for 2×2 symmetric matrices

$$B_{p,q} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left\| \begin{bmatrix} x & y \\ y & z \end{bmatrix} \right\|_{p\to q} \leq 1 \right\}$$

(even in this low dimension, computing the NP-hard balls took hours via Lemma 1 below). The balls for these open questions are printed in gold.

Operator norms measure the maximum possible magnitude of effect from an operation (roughly, something that takes inputs yielding results). Measuring the fastest a ship can turn given ideal conditions or the peak mileage an engine can achieve over all possible compositions of fuel.

Here we consider the operation of multiplying a fixed matrix A by a given input vector v. The size of effect is then measured by how large the output vector Av is relative to v. To formalize this, suppose input vectors are measured by some ℓ_p norm and output vectors are measured by some ℓ_q norm. Then the *induced* $p \to q$ *operator norm* of a matrix/operator A is

$$\|A\|_{p\to q} = \sup_{\|v\|_p \leq 1} \|Av\|_q .$$

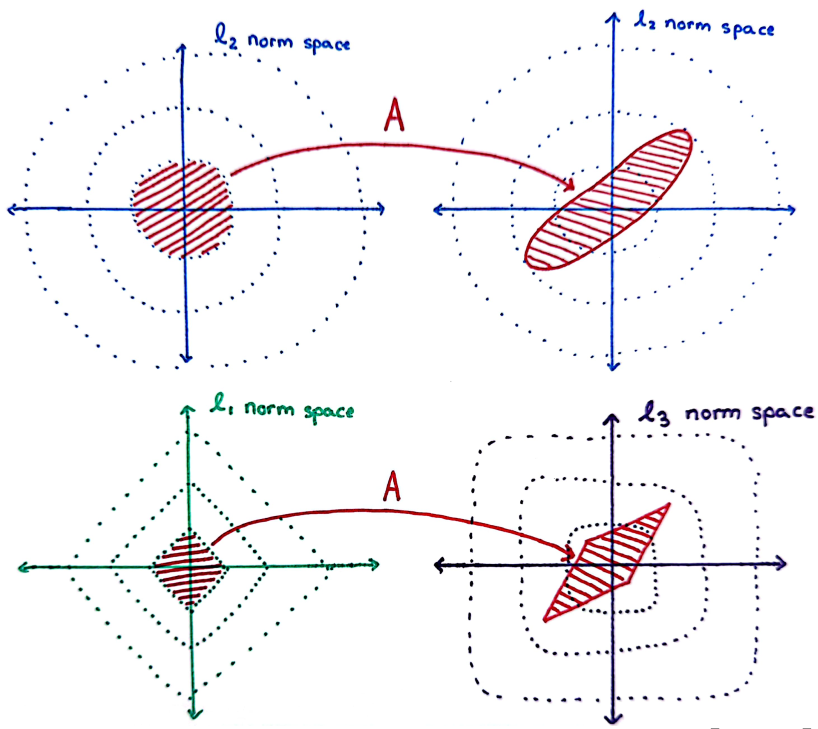

Figure 1. Example application of the matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ between various ℓ_p spaces. The level sets of points with norm 1, 2, and 3 are shown dotted for each space. The first row maps the two-norm ball into a two-normed space where we see $\|A\|_{2\to 2} = 2$. The second row maps the one-norm ball into three-normed space where $\|A\|_{1\to 3} \approx 1.782$.

The 3D Printing Source Code and Details

Disclaimer before you dive into these files: I am a mathematician professionally with only a self-taught/amateur background in three-dimensional printing and modeling.

- ❑ `.stl` design files are available at `https://www.printables.com/model/245192`
- ❑ `.nb` Mathematica file is available at `https://github.com/profgrimmer/Induced`
- ❑ This `.pdf` is available at `ams.jhu.edu/~grimmer/Induced.pdf`

The Shape of Induced Norms

Lemma 1. For any A, $\|A\|_{p\to q} = \sup_{\|x\|_p \leq 1, \|y\|_{q^*} \leq 1} y^\top Ax$.
Proof Idea. Use Hölder's Inequality.
This result enables direct computation via optimizing a quadratic over simple sets, which we used to compute the norm balls below.

Lemma 2. For any A, $\|A\|_{p\to q} = \|A^\top\|_{q^*\to p^*}$.
Proof Idea. Use the symmetry of Lemma 1.
This results in the symmetry across the diagonal, composed of the dual pairs $(1,\infty), (4/3,4), (3/2,3), (2,2), (3,3/2)(4,4/3), (\infty,1)$.

Exercises and Observations: Geometric properties below
(i) The horizontal cut through each ball corresponds to the subspace of diagonal matrices. Whenever $q \geq p$, this cut is a square.
(ii) For what p and q is this horizontal cut a perfect circle?
(iii) When $p = 1$ or $q = \infty$, there is fourfold rotational symmetry around the y-axis, otherwise it's only twofold symmetry. Why?

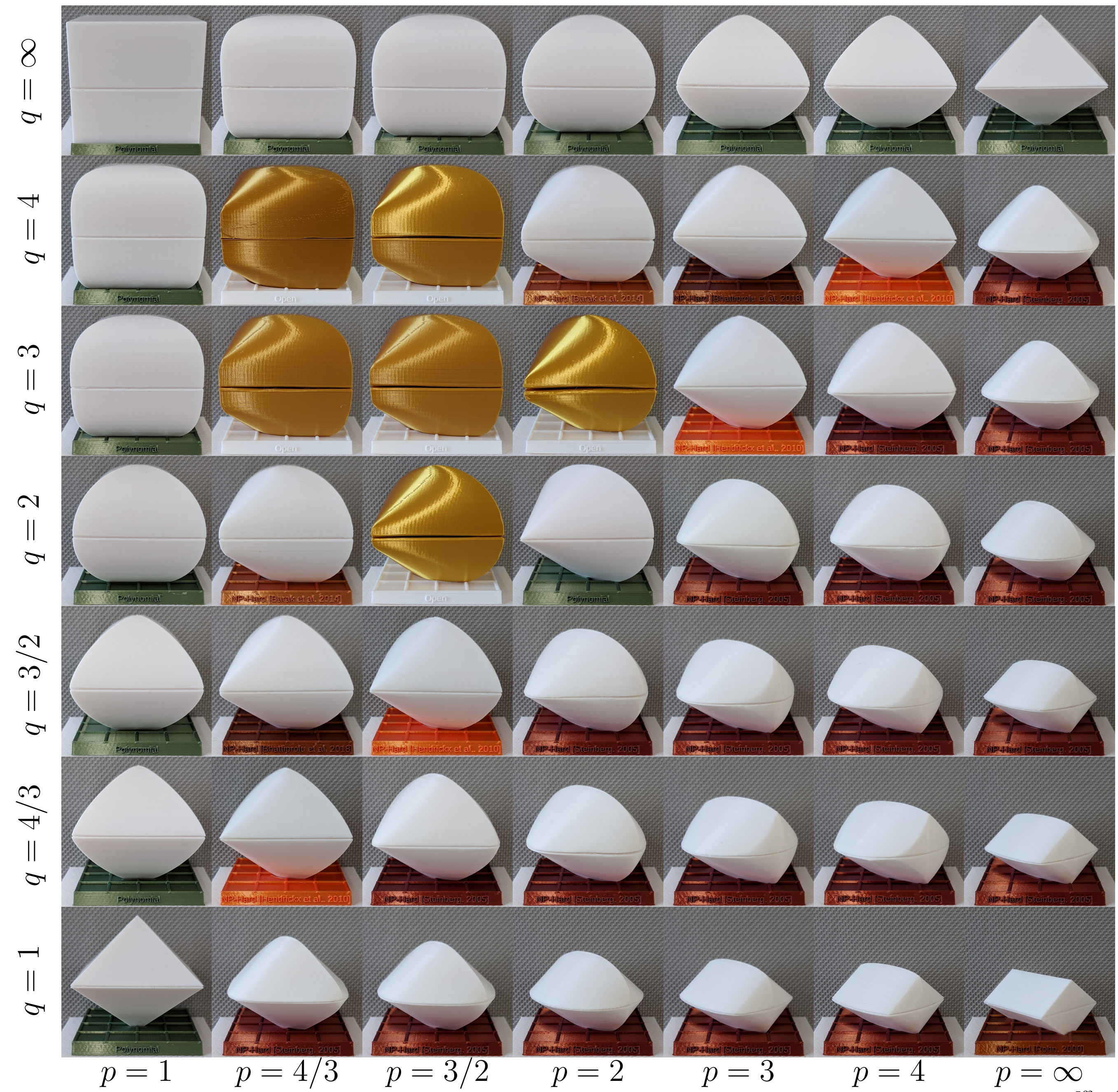

Figure 2. A collection of forty nine 2×2 symmetric $p \to q$ induced norm balls embedded in three dimensions as $(x,y,z) \mapsto \begin{bmatrix} x & y \\ y & z \end{bmatrix}$. The stand underneath each ball denotes the computational complexity of computing $\|A\|_{p\to q}$ where a *green* stand means polynomial, a *red or orange* stand denotes NP-hard, and a *white* stand (and golden ball) denotes the computational complexity is an open problem.