# Printable Topics in Mathematics

**Benjamin Grimmer**

**Johns Hopkins, AMS**

# The Grading Polyhedron

## Applications to the Administration of Courses in Higher Education (at least the ones I teach)

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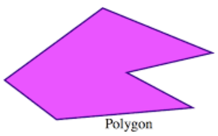
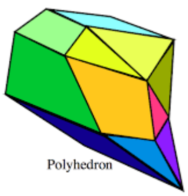
*Polygons* are shapes in two-dimensions whose boundary is given by several straight line segments. Here we consider the higher dimensional generalization of polygons, *polyhedrons* whose boundaries are higher dimensional flat faces.




For our purposes, we only consider convex bodies (unlike the examples to the left). Formally a convex polyhedron in dimension $n$ is given by the intersection of several linear inequalities:

$$a_1x_1 + \cdots + a_nx_n \leq b.$$

For example, in two dimensions, the boundary of each inequality aligns with one line segment bounding the overall polygon. In three dimensions, each inequality cuts one face of the final polyhedron. Beyond three dimensions, these are admittedly hard to visualize. Below is the three-dimensional grading polyhedron of interest.

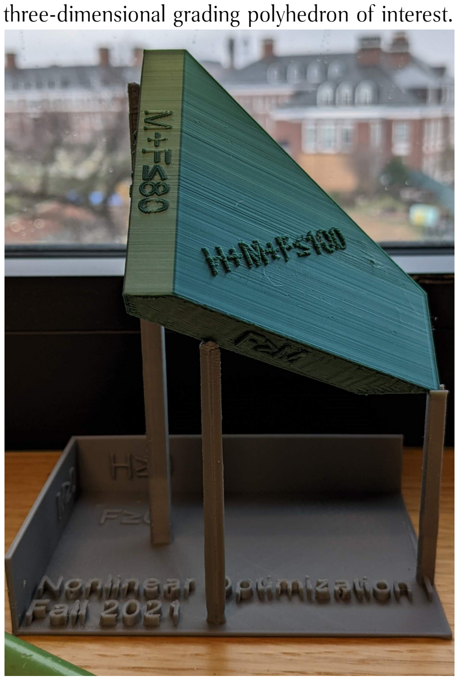



Figure 1: The Grading Polyhedron. All points $(H, M, F)$ satisfying Rules 1-4 are plotted in space with the origin $(0, 0, 0)$ at the corner of the stand. $P = 100-H-M-F$ is omitted since it is determined by the other three quantities. In total, this polyhedron has ten corners and seven faces. All faces are labeled by the corresponding inequality.

Grading rubrics and syllabuses typically seek to reward students for their individual areas of excellence while maintaining overall fairness and simplicity. This nontrivial task is central to keeping a diverse classroom motivated. Ideally, students that excel in different modalities (say, timed vs non-timed evaluations) are measured accordingly yet fairly.

### *We propose an "optimal" grading rubric.*

Consider a class with four graded components: homework worth $H$ percent of the course grade, a midterm exam worth $M$ percent, a final exam worth $F$ percent, and participation worth the remaining $P = 100 - H - M - F$ percent. Rather than fixing these values at the start of the semester (as is typically done), we will select the very best rubric individually for each student once all coursework has been graded. Hence if a student excels on the final, then that rubric portion will be raised, or if they struggled with exams, then those portions will be lowered.

*Of course, there must be limits.* The following rules limit the allowable rubrics to (in my opinion) a fair and reasonable set:

**Rule 1: Everything matters.** The three main course components should be worth at least fifteen percent. Mathematically,

$$H, M, F \geq 15\ .$$

**Rule 2: Participation's effect should be small.** It should be between zero and ten percent. Mathematically, this amounts to

$$90 \leq H + M + F \leq 100\ .$$

**Rule 3: Exams should be most but not all of the class.** The midterm and final exam weights should total between fifty and eighty percent of the overall grade. Mathematically, we require

$$50 \leq M + F \leq 80\ .$$

**Rule 4: The final is at least as important as the midterm is.** A fairly common condition. Mathematically, rubrics must have

$$M \leq F\ .$$

## *3D Model Viewer and Source File Details*

Disclaimer before you dive into these files: I am a mathematician professionally with only a self-taught/amateur background in three-dimensional printing and modeling.

- `.stl` files and 3D model viewer are available at

  `printables.com/model/239634`

- This `.pdf` is available at

  `ams.jhu.edu/~grimmer/Grading.pdf`

## The (Primal) Grading Program Details

At the end of the semester, for each student, we need to compute the best rubric satisfying the above four rules. Let's denote

$$
\begin{aligned}
C_H &= \text{Student's score in Homework Component,}\\
C_M &= \text{Student's score in Midterm Exam Component,}\\
C_F &= \text{Student's score in Final Exam Component,}\\
C_P &= \text{Student's score in Participation Component.}
\end{aligned}
$$

Their score for a given rubric comes from adding up the product of their scores and the rubric's weighting in each component

$$\frac{C_H H + C_M M + C_F F + C_P(100 - H - M - F)}{100} \ .$$

Computing the maximizing rubric is an example of a *Linear Program* since our objective function and inequality constraints are all linear functions of the variables $H, M, F$. All together,

$$
\begin{aligned}
\max\ & \frac{C_H H + C_M M + C_F F + C_P(100 - H - M - F)}{100}\\
\text{s.t. } & H + M + F \leq 100 && (1)\\
& H \geq 15 && (2)\\
& M \geq 15 && (3)\\
& F \geq M && (4)\\
& M + F \geq 50 && (5)\\
& M + F \leq 80 && (6)\\
& H + M + F \geq 90 \ . && (7)
\end{aligned}
$$

Rules 1-4 are written out here as seven linear inequalities. Note $F \geq 15$ is not included as its implied by $M \geq 15$ and $F \geq M$.

## Statistics from Previous Courses Graded

Among past instances (in JHU's Nonlinear Optimization I and II), the three most common optimal rubrics had $(H, M, F, P)$ as

$$(40, 25, 25, 10),\ (15, 37.5, 37.5, 10),\ (50, 15, 35, 0) \ .$$

Aggregating over all students, using a maximizing rubric gave an average score of $86.2$, whereas using the most average rubric gave an average score of $81.9$, and using a minimizing rubric (giving the worst justifiable grade under Rules 1-4) gave an average score of $77.2$. As a result, the effect of selecting a rubric in favor or against a student constitutes a whole letter grade (on average).

## Certificates of Optimal Grading

We conclude by grading two example students and arguing based on various combinations of the constraints that we have graded them as highly as possible. This is a taste of the deep and powerful tool central to optimization known as duality. Duality provides a convincing mechanism to show for yourself or a student that no better rubric satisfying Rules 1-4 exists.

## Example Student Grading

The grading program can be solved using algorithms like the Simplex Method or Interior Point Methods. Since our problem is of small size, we can just compute the objective value at all ten corner points and take the best one (A good exercise: Why is it good enough to just consider finding the best corner point?).

Note the ten corners of this polyhedron correspond to rubrics with $(H, M, F, P)$ as $(15, 40, 40, 5)$, $(20, 40, 40, 0)$, $(40, 25, 25, 10)$, $(50, 25, 25, 0)$, $(20, 15, 65, 0)$, $(50, 15, 35, 0)$, $(15, 37.5, 37.5, 10)$, $(15, 15, 65, 5)$, $(40, 15, 35, 10)$, and $(15, 15, 60, 10)$.

### Student Grading Example - Alice

Alice consistently worked on homework in advance and routinely asked both clarifying and exploratory questions in office hours. As a result, she managed to achieve perfect homework and participation scores, $C_H = C_P = 100$. However, the time pressure of exams and lack of ability to talk through questions out loud lead to lower scores of $C_M = C_F = 70$.

Alice's maximum course score is then $85$, given by the rubric

$$(H, M, F, P) = (40, 25, 25, 10) \ .$$

Let's verify this is the maximum under Rules 1-4: Given these course scores, Alice's objective function simplifies to maximizing

$$H + 0.7M + 0.7F + (100 - H - M - F) = 100 - 0.3(M + F) \ .$$

Since every feasible rubric satisfies equation (5), Alice's score for every rubric is at most $100 - 0.3 \times 50 = 85$. Since the rubric above achieves this $85$ score, it is optimal.

### Student Grading Example - Bob

Bob missed one early homework and generally approached them as a learning opportunity to make mistakes and get feedback, scoring $C_H = 62$ overall. Preferring not to engage in office hours or lectures, Bob got $C_P = 0$ (which is fine as we can set the weight $P = 0$). By the exams, Bob had adapted to the course's content and expectations, yielding $C_M = 81$ and $C_F = 93$

Bob's maximum course score is also $85$, given by the rubric

$$(H, M, F, P) = (20, 15, 65, 0) \ .$$

Let's verify this is the maximum under Rules 1-4: Given these course scores, Bob's objective function simplifies to maximizing

$$0.62H + 0.81M + 0.93F \ .$$

Note every feasible rubric satisfies equations (1), (3), and (6). Adding together inequality (1) times $0.62$, inequality (3) times $-0.12$ and inequality (6) times $0.31$, rather magically, we find

$$0.62H + 0.81M + 0.93F \leq 85 \ .$$

As a result, every feasible rubric assigns Bob a grade of at most $85$ and the rubric above is optimal. (These "magically" chosen multipliers are known as dual variables or Lagrange multipliers.)