Printable Topics in Mathematics

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Function p-Norm Balls

A Generalization of p-Norms to (Quadratic) Functions

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CALCULUS and computational mathematics often quantify small changes of functions throughout numerical processes. Previously (`ams.jhu.edu/~grimmer/pNorm.pdf`), we considered the p-norm of a vector v for $p \geq 1$ as

$$\|v\|_p = \left(\sum |v_i|^p\right)^{1/p} .$$

Here we illustrate their natural extension to function norms: The *functional* p*-norm* of some $f\colon [-1,1] \to \mathbb{R}$ for $p \geq 1$ is

$$\|f\|_p = \left(\int_{-1}^{1} |f(t)|^p \ \mathrm{dt}\right)^{1/p} .$$

An Orthonormal Basis for Functions

A needed tool, inner products, provide a notion of angles, measuring how similar two functions are. We define the inner product between functions $f, g\colon [-1,1] \to \mathbb{R}$ as

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t) \ \mathrm{dt} .$$

Two functions are orthogonal if their inner product is zero. A famous family of orthogonal polynomials are Legendre's:

$$P_0(t) = 1 \qquad P_3(t) = \frac{1}{2}(5t^3 - 3t)$$

$$P_1(t) = t \qquad P_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3)$$

$$P_2(t) = \frac{1}{2}(3t^2 - 1) \qquad P_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t)$$

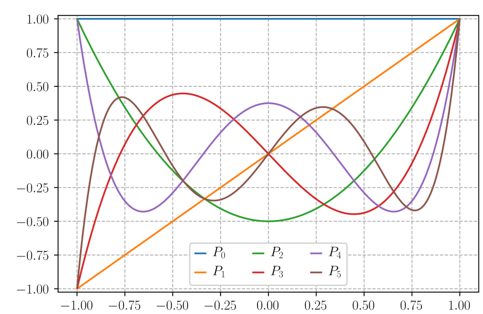

Figure 1. The first six Legendre polynomials.

3D Model Viewer and Links

- ❑ `.stl` files and 3D model viewer at `printables.com/model/826063`
- ❑ This `.pdf` is available at `ams.jhu.edu/~grimmer/FuncNorm.pdf`

Embedding Functions in 3D for Printing

In general, it is tricky to visualize a space of functions as it is "infinite-dimensional". To do so in 3D, we need a family of functions with exactly three degrees of freedom. Quadratics will do...

$$(x, y, z) \mapsto Q_{x,y,z}(t) = x + yt + zt^2 .$$

Although the choice above of having x, y, z the coefficients is nice, it does not provide an "orthonormal basis". To get that, we instead add together Legendre polynomials...

$$(x, y, z) \mapsto Q_{x,y,z}(t) = xP_0(t) + yP_1(t) + zP_2(t)$$

which can also represent every degree two polynomial. The benefit of selecting such a basis is that $\|Q_{x,y,z}\|_2 \leq 1$ if and only if $\|(x, y, z)\|_2 \leq 1$. Hence the ball of unit size functions $\|Q_{x,y,z}\|_2 \leq 1$ is just a ball in the regular sense. For $p \neq 2$, the shape of the unit ball goes a bit wild. See below :)

Figure 2. From left to right, Function p-norm balls of quadratic (in Legendre's basis) with $p = 1, 4/3, 3/2, 2, 3, 4, \infty$, defined as the set $\{(x, y, z) \mid \|Q_{x,y,z}\|_p \leq 1\}$. For $p < 1$, these are no longer norms (Exercise: Geometrically, what changes in the shape?).