Printable Topics in Mathematics

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A Collection of CVaR Norm Balls

Norms and Measures of Magnitude Focusing on the Most Extreme Values in a Data Set

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CONDITIONAL VALUE AT RISK (CVaR) measures, commonly used in finance, capture the tail risk of an investment, measuring the expected losses in the, say, worst p fraction of cases. This is useful in portfolio optimization and risk management applications when planning against shortfalls.

The idea of measuring just the most extreme entries can be understood visually in terms of lifting vector norms to higher dimensions. Suppose you have possible m outcomes of an investment each costing $x_i \geq 0$ dollars. The average (expected) cost of this investment is then

$$\frac{1}{m}\sum_{i=1}^{m} x_i = \frac{1}{m}\|x\|_1 \ .$$

Sorting the outcomes as $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(m)}$, we can express the average (expected) cost in the worst p fraction of cases as (using the norm lifting notation defined to the left)

$$\frac{1}{pm}\sum_{i=(1-p)m}^{n} x_i = \frac{1}{pm}\|x\|_{mp,\ \|\cdot\|_1} \ .$$

One Mechanism For Norm Lifting

Consider any norm $\|\cdot\|$ defined in $\mathbb{R}^n$ (recall being a norm means satisfying the Triangle Inequality, Positive Definiteness, and Absolute Homogeneity). For any $m > n$, we can define a new function by taking the norm of the n largest magnitude entries: Given $x \in \mathbb{R}^m$, define

$$\|x\|_{n,\|\cdot\|} := \|(x_{\sigma(m-n+1)}, \dots x_{\sigma(m)})\|$$

where σ has $|x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \cdots \leq |x_{\sigma(m)}|$. This operation takes the lower dimensional norm of the top magnitude n entries in x. Rather surprisingly, this is always a norm in $\mathbb{R}^m$ (proving this is a good exercise!).

3D Model Viewer and Links

- `.stl` files and 3D model viewer at `printables.com/model/358705`
- This `.pdf` is available at `ams.jhu.edu/~grimmer/CVaR.pdf`

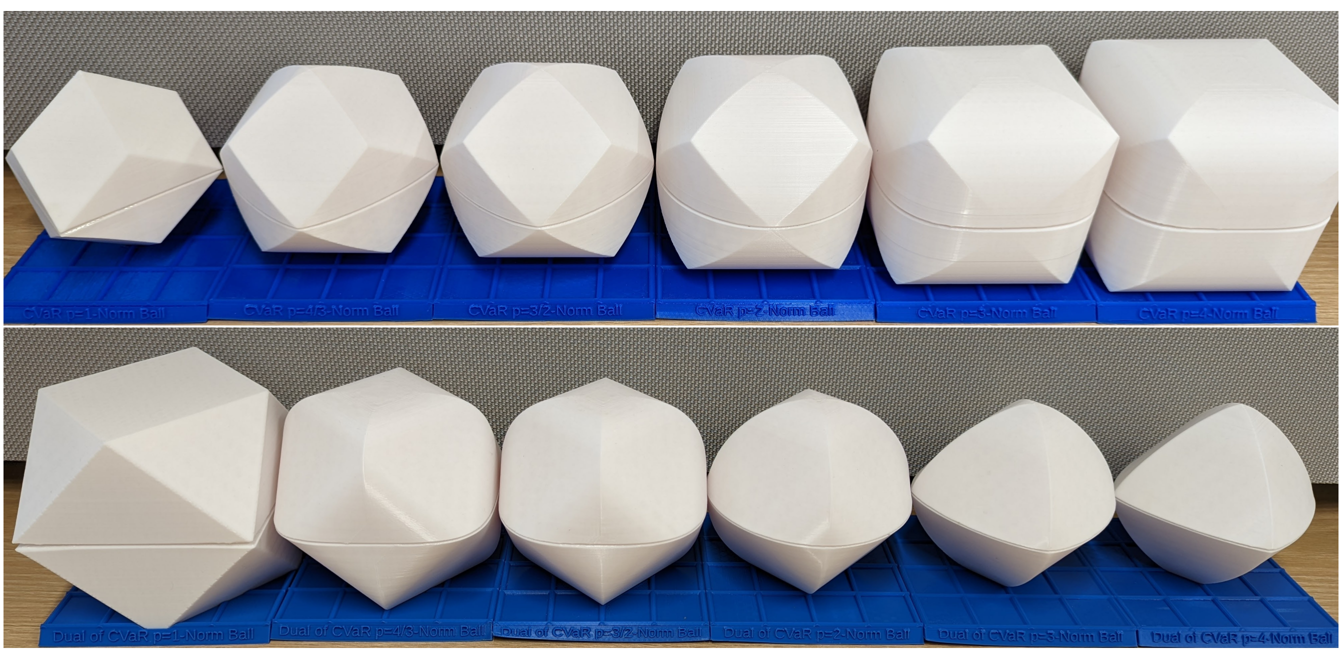

Figure 1. The first row shows several "CVaR" norm balls given by taking increasing p-norms of the top two magnitude entries in (x, y, z), where the p-norm is $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. The first of these with $p = 1$ aligns with the classic CVaR expected value. The $p = 2$ ball has the novel property that it rolls smoothly only in coordinate basis directions (why?). The second row shows the dual of each of these norms (recall, given a norm $\|v\|$ for vectors v, we can construct its *dual norm* $\|w\|_*$ as $\|w\|_* = \max_{\|v\|=1} w^\top v$.)