Printable Topics in Mathematics

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The Hidden 2-Norm in 4-Norms

A Puzzle with Applications to Second-Order and Fourth-Order Programs

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Reduction between Optimization Problems

Using p-norms can provide a useful computationally tractable family of optimization problems. Optimization problems with constraints $\|Ax-b\|_2 \leq c^T x+d$ are called second-order cone programs. The geometric property shown here shows any two-norm constraint can be reformulated as a four-norm constraint $\|Ax-b\|_4 \leq c^T x+d$ (and an equality constraint to enforce the cross-section).

Figure 2. A 4-norm ball living in 3D with a 2-norm ball living inside on a carefully cut 2D cross section of it.

3D Model Viewer and Links

- `.stl` files and 3D model viewer at `printables.com/model/553000`
- This `.pdf` is available at `ams.jhu.edu/~grimmer/2in4Norm.pdf`

P-NORMS are widely used through mathematical modeling. These were previously showcased in the prior 3D prints (see `ams.jhu.edu/~grimmer/pNorm.pdf`). That writeup gives a more complete picture of these fundamental objects. To recap, we define the p-norm of a vector v as

$$\|v\|_p = \left(\sum |v_i|^p\right)^{1/p} .$$

In 2D and 3D, we can visualize these by considering the set of all points with norm at most one as shown below in 2D and 3D.

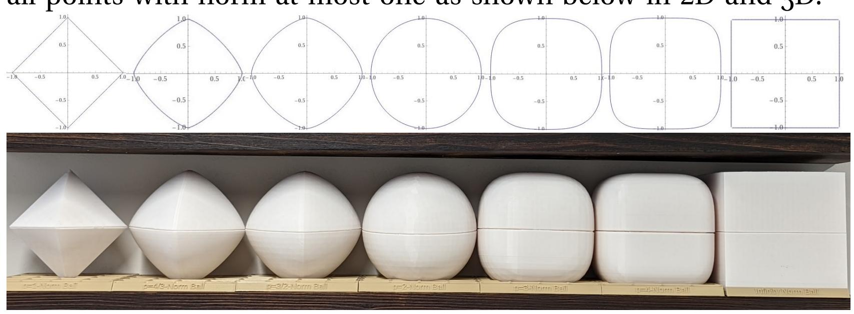

Figure 1. Unit balls $p = 1, 4/3, 3/2, 2, 3, 4, \infty$. Cutting each 3D ball horizontally has cross-section exactly the related 2D ball.

Surprisingly, these horizontal cross-sections (where a lower dimensional p-norm lives inside a higher dimensional p-norm) are not the only interesting cuts! If you carefully (diagonally) cut a 4-norm ball, you find a 2-norm ball inside! See Figure 2!

The Exercise for the Reader

Explain why a 2-norm ball lives inside the 4-norm ball.
(This question was first posed to me by Pablo Parrilo)

Two hallmarks of an excellent explanation of this phenomena:
(i) This idea can work in higher dimensions, so an ideal explanation wouldn't be particular about cutting in 3D to a 2D ball.
(ii) This idea can extend to other values of p. Reasonable conjectures would be that a 6-norm or 8-norm might have a 4-norm living inside it (and consequently a 2-norm living inside that interior norm). There could be a whole *hierarchy* of structure here to explore...