A generalization of Cesàro’s theorem

We will need the following generalization of Cesàro’s theorem.

Claim 1 If real numbers \(x, x_1, x_2, \ldots\) satisfy \(x_n \to x\), and if \(b_n \uparrow \infty\), then, with \(b_0 := 0\),

\[
\frac{1}{b_n} \sum_{m=1}^{n} (b_m - b_{m-1})x_m \to x \quad \text{as} \quad n \to \infty.
\]

Proof. Given \(\epsilon > 0\), choose \(N(\epsilon)\) so that \(m > N(\epsilon)\) implies \(|x_m - x| < \epsilon\). Also, since the sequence \((x_n)\) is convergent, we can bound \(|x_m - x| \leq M\) for all \(m\). Let \(n \geq N \equiv N(\epsilon)\). Then

\[
\left| \sum_{m=1}^{n} (b_m - b_{m-1})x_m - b_n x \right| = \left| \sum_{m=1}^{n} (b_m - b_{m-1})(x_m - x) \right|
\]

\[
\leq M \sum_{m=1}^{N} (b_m - b_{m-1}) + \epsilon \sum_{m=N+1}^{n} (b_m - b_{m-1})
\]

\[
= Mb_N + \epsilon(b_n - b_N).
\]

Divide by \(b_n\) and let \(n \to \infty\) to find

\[
\limsup_{n \to \infty} \left| \frac{1}{b_n} \sum_{m=1}^{n} (b_m - b_{m-1})x_m - x \right| \leq \epsilon.
\]

Since \(\epsilon\) is arbitrary, we are finished. \(\square\)