(c) fij's agree for X and embedded chain.

Conseq. of (c):
Every thing from discrete-time embedded MC involving only fij's carries over to X, including decomp. of the state space into recurs. & trans. comm. classes.
Preliminaries...

There is no analogue of periodicity for cont.-time MCs:

THEM. \( \forall i, \forall j \), either

\[
\delta_{ij} + f_{ij} = 0 \quad \text{and} \quad \forall 0 < t < \infty, P_{ij}(t) = 0
\]

or

\[
\delta_{ij} + f_{ij} > 0 \quad \text{and} \quad \forall 0 < t < \infty, P_{ij}(t) > 0
\]

Proof: Exercise: It isn't hard and isn't so interesting.
Mean return times, post null recurrence

**DEFN.** The mean return time \( m_i \) to a state \( i \) is defined by

\[
  m_i := E_i T_i
\]

A nonabsorbing recurrent state \( i \) is called positive recurrent or null recurrent according as

\[
m_i < \infty \text{ or } m_i = \infty.
\]
An absorbing state $i$ is defined to be positive recurrent (even though $m_i = \infty$ in this case).

**EXER.** Give examples where $X$ is pos. recurrent & embedded chain is null recurrent & via versus.
Problems

Occupation times

**DEFN.** Write

\[ \tau_t(j) := \int_{s=0}^{t} \mathbb{I}\{X(s) = j\} \, ds \]

for the amount of time in \([0,t]\) the chain \(X\) spends in state \(j\). Similarly,

\[ \tau_j := \int_{s=0}^{\infty} \mathbb{I}\{X(s) = j\} \, ds \]

which is

\[ \lim_{t \to \infty} \tau_t(j). \]
Limiting behavior of $T_t(s)/t$ and its expectation

CASE 1. $j$ is transient. Then

$$0 \leq E_T(s) = \frac{\mathbb{P}(T_j < \infty)}{v_j (\lambda - f_j s)} < \infty$$

and $T_t(s) \uparrow T(s) < \infty$ as $t \uparrow \infty$ wp1

implies $T_t(s)/t \rightarrow 0$ wp1,

which implies $E(T_t(s)/t) \rightarrow 0$ by the BCT.
CASE 2. \( j \) is recurrent.

(a) \( j \) is non-absorbing.

Then \( \tau(j) = \infty \times I_{\{T_j < \infty\}} \)
(\text{where} \( \infty \times 0 := 0 \)) \text{ and so}

\[ E(T_j) = \infty \times P(T_j < \infty). \]

Moreover,

\[ \frac{\tau(j)}{t} \rightarrow \frac{I_{\{T_j < \infty\}}}{\frac{1}{\nu_j m_j}} \]

which implies (by BCT) \( \text{wp} \)

\[ E\left( \frac{\tau(j)}{t} \right) \rightarrow \frac{P(T_j < \infty)}{\nu_j m_j} \]
(b) \( j \) is absorbing.

Then \( \tau(j) = \infty \times I_{\{T_j < \infty\}} \)

and so

\[ E_\tau(j) = \infty \times P\{T_j < \infty\} \]

Moreover,

\[ \frac{\tau_k(j)}{t} \Rightarrow I_{\{T_j < \infty\}} \]

which implies (by BCT)

\[ E\left(\frac{\tau_k(j)}{t}\right) \Rightarrow P\{T_j < \infty\} \]
Prelims... 

Positive recurrent, null recurrent, 
& transience are Comm. 
Class properties

**Lemma.** A state \( j \) is transient if \( \int_0^\infty P_{jj}(s) \, ds < \infty \)

positive recurrence if 
\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t P_{jj}(s) \, ds > 0.
\]
DEFN. A pmf $\pi$ is stationary if
\[
\pi_0 = \pi \Rightarrow \pi_t = \pi P(t) = \pi \quad \forall t \geq 0,
\]
that is, if $(\pi_t)$
\[
\sum_i \pi_i P_{ij}(t) = \pi_j \quad \forall j \quad \forall t \geq 0.
\]

THM. A pmf $\pi$ is a stat. dist. if
\[
\sum_i \pi_i r_{ij} = 0 \quad \forall j.
\]
Proof of them for finite state space

If \( \pi = \pi P(t) \)
then \( \Theta = \pi P'(t) \)
\[ (t=0) \Rightarrow O = \pi G. \]

Conversely, if \( \pi G = 0 \),
then \( \pi GP(t) = 0 \ \forall t \),
which by (BD) implies
\[ \frac{d}{dt} [\pi P(t)] = \pi P'(t) = 0 \ \forall t \]
\[ \Rightarrow \pi P(t) = \pi P(0) = \pi. \]
OUTLINE of proof of Thm for finite or stably infinite State Space:

1) \[ \text{stat.} \Rightarrow \Pi G = 0 \]
   Expand \( P_{ij}(t) \) in \( \Pi P(t) = \Pi \) using \( (F I) \), then let \( t \to \infty \).

2) \[ \Pi G = 0 \Rightarrow \text{stat.} \]
   Rewrite \( \Pi G = 0 \) in the form
   \[ \pi_j \varphi_j = \sum_{i \neq j} \pi_i \varphi_i \]
Using (FR), argue by induction on $N$ that

$$\prod_{j} j \geq \sum_{i} \prod_{j} P^{(N)}_{ij}(t)$$

for all $t \geq 0$ and $N \geq 0$.

Let $N \to \infty$ to find, via MCT,

$$\prod_{j} j \geq \sum_{i} \prod_{j} P^{(\infty)}_{ij}(t).$$

By summing over $j$, deduce that equality holds.
STEADY STATE

Yadda, yadda, yadda

The unique stationary distribution $\pi$ for an irreducible, positive recurrent, continuous-time MC is also steady state.

IRRED. B&D CHAINS

Look at embedded chain to get

RESULT:

Transient if $\sum_{i} \frac{\mu_i}{\lambda_i} > 0$
Irred. B2D proc.:
Pos. vs. null recurrence
and stationarity

Step 1. Show using $\pi G = 0$
that a pmf $\pi$ is stat. If

$$\pi_i = \pi_i \pi_0, \quad i \geq 0,$$

with

$$\pi_i := \begin{cases} 1 & \text{if } i = 0 \\ \frac{\lambda_i \pi_{i-1}}{\sum \lambda_i \pi_i} & \text{if } i > 1 \end{cases}$$

Step 2. Show that if $\sum \pi_i < \infty$,

then

$$\pi_i := \frac{\pi_i}{\sum_{k=0}^{\infty} \pi_k}, \quad i \geq 0$$
defines the unique stat. dist

**Step 3.** Show that if \( \sum p_i = \alpha \neq 0 \text{ then no stat. dist} \) exists.

**Classification of irreducible B&BD processes:**

- **Pos. rec.** if \( \sum \frac{m_i}{M} < \alpha \)
- **Trans.** if \( \sum \frac{m_i}{M} \geq \alpha \)
- **Null rec.** if both series = \( \alpha \)

**Note.** If state space is finite, proc. if pos. rec. & formula works.
APPLIES: $\pi$-state MC

$$\pi = \begin{pmatrix} \frac{\lambda}{\lambda + \mu} & \frac{\lambda}{\lambda + \mu} + \mu \\ \mu & \frac{\lambda}{\lambda + \mu} \end{pmatrix}$$

CHECK using BDP results.

#2) Infinite-server queue

$\lambda_i = \lambda > 0$, $\mu_i = i\mu$ for $i \geq 0$
Check using B&D results that proc. is recurrent.

Moreover, \( P_i = \frac{(\lambda / \mu)^i}{i!}, i \geq 0 \),

so proc. is pos. recurrent, and

\( \pi_i = \text{Poi}(\lambda / \mu) \)

is unique stat. & steady state dist.
APPLIC 3.

N server queue. \((m/m/n)\)

\[ \lambda_i = \lambda \geq 0 \quad \forall i \geq 0 \]

\[ \mu_i = \sum_{i \geq n} \mu_i \quad 0 \leq i < N \]

\((\mu > 0)\).

Show that this proc. is

\[ \begin{cases} \text{transient} \\ \text{null recurr.} \\ \text{acc as } N \mu \end{cases} \]

(\text{post. recurr.})