1.) Show that any solution in $[0, 1]$ to $(FD)$
is a solution to $(FI)$; in short, that $(FD) \implies (FI)$.

**Details:** Suppose $0 \leq P^* \leq 1$ satisfies $(FD)$.
Then $e^{q_j t} P^*_j (t) - S_{ij}$ has derivative

$$e^{q_j t} \sum_{k \neq j} P^*_k (t) \delta_{kj} \in [0, \infty)$$

and so (e.g., by the mean value theorem)
increases from 0 to $e^{q_j t} P^*_j (t) - S_{ij}$
$\leq e^{q_j t_0} < \infty$ as $t$ increases from 0
to any given $t < \infty$. 
Next we use Theorem 31.2 from Billingsley's *Probability and Measure*:

A nondecreasing function $F$ is differentiable except on a set of Lebesgue measure 0. The derivative $F'$ is measurable and nonnegative, and satisfies

$$\int_{t=a}^{b} F'(t) \, dt \leq F(b) - F(a)$$

for all $a \leq b$.

to conclude that

$$0 \leq \int_{s=0}^{t} e^{\mathbf{z}^*s} \sum_{k \neq j} \mathbf{P}^{*k}_{ik}(s) \mathbf{e}_k \cdot \mathbf{e}_j \leq e^{\mathbf{z}^*t} < \infty.$$

But now we can use Thm. 8.21 in Rudin's *Real and Complex Analysis*:
Suppose $f$ is a real-valued function on $[a, b]$ which is differentiable at every point of $[a, b]$, and assume that $|f'|$ is integrable on $[a, b]$ (i.e., that $f' \in L^1$ on $[a, b]$). Then

$$f(x) - f(a) = \int_a^x f'(t) \, dt \quad (a \leq x \leq b).$$

Thus $P^*$ satisfies (FI).

2.) Show that $P$ satisfies (FD).

**DETAILS.** For this we show that $\sum_{k,j} P_{ik}(t) z_{kj}$ is a finite continuous function of $t \geq 0$; then (FD) follows for $P$ by applying the FTOC to $(e^{g_it} \text{ times})$ (FI) for $P$.

Indeed, ...
Indeed, by the BACKWARD integral equation

\[(BI) \quad P_{ikj}(t) = S_{ij} e^{-g_{i}t} + \int_{r=0}^{t} (\sum_{k \neq i} P_{ikj}(r)) e^{r_{i}t} dr\]

for \( P \), we have

\[(*) \quad \sum_{h \neq j} P_{ikj}(t) q_{kj} = e^{-g_{i}t} \left[(1-S_{ij})q_{ij} + g_{i} \int_{0}^{t} \sum_{h \neq j} (\sum_{k \neq i} P_{ikj}(r)) q_{kj} e^{r_{i}t} dr\right],\]

which implies that

\[e^{g_{i}t} \sum_{h \neq j} P_{ikj}(t) q_{kj}\]

increases with \( t \). Hence if \[\sum_{k \neq j} P_{ikj}(t) q_{kj} = \infty\]

for \( t = t_0 \), then the same is true for all \( t \geq t_0 \).
2. $P$ satisfies (FD).

But then (FI) for $P$ gives $P_{xy}(t) = \infty$ for all $t > t_0$, which is absurd. So

$$\sum_{h \neq j} P_{ik}(t) q_{kj} < \infty \text{ for all } t \geq 0.$$ 

But then (4) on page 247.4 above implies [by the same argument as for Step 1 of (BI) $\rightarrow$ (BD)] that

$$\sum_{h \neq j} P_{ik}(t) q_{kj}$$

is continuous in $t \geq 0$, which is what we needed to show.

THE THEOREM ON PAGE 242 ABOVE IS PROVED.