(a) \( n_i \)'s odd above.

(in particular, it's suff. that
state space is finite,
and so are all \( n_i \)'s)

(b) \( (X(\tau_n)) \) is recurrent

(c) \( \sum_{i=1}^{\infty} \frac{1}{n_i} = \infty \)

and \( \prod_{i=1}^{\infty} P_i \{ X(\tau_n) \text{ visits each state at least once} \} = 1 \)

\( \forall i \).
For BkD chains, here's how things turn out:

(Assume birth rates $\lambda_i > 0$ ($i \geq 0$) and death rates $\mu_i \geq 0$ ($i \geq 0$).)

If $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} \times \frac{1}{\mu_i}$ diverges, then these rates are regular.

$$\sum_{i=0}^{\infty} \frac{\lambda_i \cdots \lambda_{i-j+1}}{\mu_i \cdots \mu_{i-j+1}}$$
NEXT:

Investigate methods for calculating \( P(t) \) from \( (a_{ij}) \). All will show that the answer to Q2 on p. 236 is YES.

**METHODS:**

- **BACKWARD** recursion formulas, S eqns., diff. eqns.
- **FORWARD** recursion formulas, S eqns., diff. eqns.
**Notation:**

\[ P_{n}^{(N)}(t) := P_{i} \{ X(t) = j \} \text{ at most } N \text{ jumps in } [0, t] \]

**Backward Recursion Formula**

**Idea:** Derive equations by considering **first** jump in \([0, t]\), i.e., jump furthest back in time.
EXECUTION:

BRFs:

\[
P_{ij}^{(N)}(t) = \sum_{k \neq i} P_{ij}^{(k)}(t) e^{-\alpha t} \]

\[
+ \int_{r=0}^{t} \left[ \sum_{k \neq i} q_{ki} P_{kj}^{(N-1)}(r) \right] e^{-\alpha (t-r)} \, dr
\]

for \( N \geq 1 \) and of course:

\[
P_{ij}^{(0)}(t) = \delta_{ij} e^{-\alpha t}
\]
Proof.

Assume $i$ is nonabsorbing. Then

$$ P_{ij}(t) = S_{ij} e^{-\nu_i t} $$

$$ + \sum_{s=0}^{t} \int_{0}^{s} (q_{ij} e^{-\nu_i s} ds) P_{ik} $$

$$ \times P_{kj}^{(N-1)} \left( t - s \right) $$

$$ = \delta_{ij} e^{-\nu_i t} $$

$$ + \sum_{r=0}^{t} \left[ \sum_{k \neq i} q_{ik} P_{kj}^{(N-1)} \right] e^{-\nu_i (t-r)} dr. $$
HW #10
not due until Mon. Apr. 30

Ross 5.15
5.21
6.2
6.3
6.6
6.9
6.13

READ Chap. 6
(Concentrate on 6.1-6.3)

READ §§8.1-8.4
An example of use of BRFs:

Pure birth process with constant birth rate on state space \{0, 1, 2, \ldots\} 

\[ \text{Poisson Process!} \]

with intensity \( \lambda \)

Exercise. Fix this:

(a) By induction, prove 
\[ P_{ij}(t) = \begin{cases} \lambda t e^{-\lambda t} & \text{if } j-i \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
(b) Pass to limit as \( N \to \infty \) to get

\[ P_{ij}(t) = \begin{cases} (\lambda t)^{i-1} \frac{e^{-\lambda t}}{(i-1)!} & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases} \]

which is t.f. for a Poi proc.
BACKWARD INTEGRAL EQNS

Pass to limit as $N \to \infty$ in (BR) to find, via BCT (or MCT),

B. I. EQUATIONS:

\[(BI) \quad P_{ij}(t) = \delta_{ij} e^{-\lambda_{ij}t} + \int_{t=}^{t} \left( \sum_{k+i} q_{ik} P_{kj}(r) \right) e^{-\lambda_{ij} (t-r)} dr \]
THM. The BI eqns have no solution other than $(P_{ij}(t))$ that is stochastic for each $i$ and $t$.

**Proof.** We need only show uniqueness. Suppose $A_i A_j A_k$ 

(A) $P_{ij}(t) = 8i e^{-\beta t} + \int_{t=0}^{t} \sum_{k,l} q_{ik} e^{-\beta (t-r)} dr$

and
(B) \( P_{ij}^{*} (t) \geq 0 \)

and

(c) \( \sum P_{ij}^{*} (t) = 1 \).

From (A) and (B),

\[ P_{ij}^{*} (t) = \delta_{ij} e^{-q_i t} = P_{ij}^{(0)} (t) \]

By (A) and (B) and induction

\[ P_{ij}^{*} (t) \geq P_{ij}^{(N)} (t) \quad \forall 0 \leq N < \infty \]

Pass to limit as \( N \to \infty \):

\[ P_{ij}^{*} (t) \geq P_{ij} (t) \]

OTHER HALF IS FREE!
By (c), \[ \sum P_{ij}(t) = 1 = \sum P_{ij}(t). \]

Therefore, \( P_{ij}(t) \equiv P_{ij}(t) \), as claimed.

**EXERCISE. Use (BI)**

to solve for t.f. of a pure birth process w/ const. birth rate.
BACKWARD
DIFFERENTIAL EQUATIONS.

IDEA: By differentiation, (BI) can be transformed into an equivalent system of diff. eqns. (BD), and so (BD) has the unique solution (Pij(t)).

EXECUTION:
If we assume that $0 \leq \hat{P}_{ij}^{*}(t) \leq 1$ for all $i, j, t$, then (B1) and (BD) are equivalent for $(P_{ij}^{*}(0))$, where

$P_{ij}^{*}(t) = \sum_{k} q_{ik} \hat{P}_{kj}^{*}(t)$ for $t > 0$

and $P_{ij}(0) = \delta_{ij}$

where by $q_{ii}$ I mean $q_{ii} = -\chi_{i}$
I talked through the proof in class.

**FORWARD METHODS**

Forward recursion formulas:

\[(FR)\quad P_{ij}(t) = \delta_{ij} e^{-\lambda_j t} + \int_0^t e^{-(\lambda_j - \lambda_k) (t-s)} P_{ik}(s) P_{kj}(s) ds \]

for \( N \geq 1 \),

and \( P(i,j)(t) = \delta_{ij} e^{-\lambda_i t} \).
(FI) \[ P_{ij}(t) = S_{ij} e^{-\lambda t} + \int_0^t \left( \sum_k P_{ik}(s) q_{kj} \right) e^{-\lambda (t-s)} ds \]

**THEM**. The FI eqns. have no sol'n \( \theta_{ij}(t) \) that is stochastic \( \forall t \).

Concerning transfer between FI & FD,
THM. The t.f. \((P_{ij}(t))\)
is the unique stochastic solution to

\[
(PD) \quad P_{ij}'(t) = \sum_k P_{ik}(t) \Delta_{kj}
\]
for \(t \geq 0\), and \(P_{ij}(0) = \delta_{ij}\).

Sketch of proof.

1.) Any sol'n in \([0, \infty)\) to \((PD)\)
is a sol'n to \((FI)\).

2.) \((P(t))\) does satisfy \((FD)\).
A rather interesting feature of the detailed proof of Step 2.) here is that it uses the (FI) and -- to settle a technical matter -- (BI)!!
Stationarity and Long-run Behavior of Continuous-Time Markov Chains

Preliminaries:
Hit and times, hitting probabilities, communicating classes, recurrent, transient
DEFNS.

**Hitting time:**
\[ T_{ij} := \inf \{ t \leq t < \infty : X(t) = j \} \]

**Hitting probability:**
\[ f_{ij} := \begin{cases} \Pi_{ij} & \text{if } i \text{ is non-abs.} \\ \delta_{ij} & \text{if } i \text{ is abs.} \end{cases} \]

FACTS. (a) A hitting time is a stopping time.
(b) \((T_j \text{ for } X) \neq (T_j \text{ for embedded chain})\),
but \( T_j(X) = \frac{1}{T_j(\text{embedded})} \).