HITTING TIMES:

Sample calc

\[ P^{n}(i, j) = \sum_{m=i}^{n} P_{c}^{i} \mathbb{1}_{T_{j}=m} P_{c}^{m,j} \]

\[ f_{ij}^{n} = P_{c} \mathbb{1}_{T_{j}=n} \]

\[ f_{ij} = \sum_{1 \leq n < \infty} f_{ij}^{n} = P_{c} \mathbb{1}_{T_{j} < \infty} \]

\[ f_{ij} = ?? \]

It turns out that the special case \( i=j \) is especially useful.
DEFN. (a) A state $i$ is said to be recurrent if $f_{ii} = 1$.

(b) $i$ is transient if $f_{ii} < 1$.

$f_{ij} = ?\?\$

key: analyze $\# \text{of visits to } j$

Idea: $T_i < \infty$ if $\# N_j \geq 1$

where $N_j = \sum_{k=1}^{\infty} I_{X_k = j}$
Program:

Compute $P_i \{ N_j \geq m \}$ in terms of $f$'s.

Also, use result to compute $E_i N_j$.

Step 1. $P_i \{ N_j \geq 1 \} = f_{ij}^+$.

Step 2. $P_i \{ N_j \geq m \} = f_{ij}^+ f_{ij}^{m-1}$

(restart)

$v_{m \geq 1}$

We've found tail d.f. for r.v. $N_j$ (under $P_i$).
What is defect 
\( (= P_i\{N_j=\infty\}) \)?

**Step 3.** \( P_i\{N_j=\infty\} = \begin{cases} f_{ij} & \text{if } j \text{ is recurrent} \\ 0 & \text{otherwise} \end{cases} \)

**Step 4.** \( P_j\{N_j=\infty\} = \begin{cases} 1 & \text{if } j \text{ is recurrent} \\ 0 & \text{otherwise} \end{cases} \)

**Step 5.** Compute \( E_iN_j \)

(a) \( j \) recurrent

(i) If \( f_{ij} > 0 \), then of course \( E_iN_j = \infty \) (by Step 3)

(ii) If \( f_{ij} = 0 \), then of course \( E_iN_j = 0 \) (by Step 1)
(b) \( \tilde{\text{Transient}} \)

\[
E_i N_j = \sum_{m=1}^{8} \mathbb{P}_i \{N_j \geq m\} = \sum_{m=1}^{8} f_{ij} f_{jj}^{m-1}
\]

by Step 2.

\[
= \frac{f_{ij}}{1 - f_{jj}}
\]

\[
< 8.
\]

Step 6. \( E_j N_j = \begin{cases} 
8 & \text{if } j \text{ is transient} \\
< \infty & \text{if } j \text{ is transient}
\end{cases}
\)
No MC with finite state space is transient (i.e., has all states transient).

\[ i \text{ recurrent } i \leftrightarrow j \]
\[ f_{ij} = 1 \land f_{ji} = 1 \Rightarrow f_{jj} = 1 \]

**Cor. 4.2.4** If \( i \) is recurrent and \( i \leftrightarrow j \) (meaning \( f_{ij} > 0 \) and \( f_{ji} > 0 \)), then \( j \) is recurrent.
Prop 4.2.1
Communication is an equivalence relation.
Cor 4.2.4 Recurrence is a communication class property.

(Also true for transience)

Example.

\[ P = \begin{bmatrix}
0 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/4 & 0 & 0 \\
0 & 1/4 & 0 & 1/3 & 1/15 \\
0 & 0 & 1/3 & 0 & 1/15 \\
0 & 0 & 0 & 1/15 & 0
\end{bmatrix} \]
1-step transition diagram

\[ 0 \Rightarrow 1 \Rightarrow 5 \Rightarrow 4 \Rightarrow 2 \Rightarrow 3 \Rightarrow 0 \]

\[ [0] = \{0, 1\} \quad \text{recurrent} \]
\[ [2] = \{2, 4\} \quad \text{recurrent} \]
\[ [3] = \{3, 5\} \quad \text{transient} \]
KNOW THIS!

- If $MC$ begins in recurrent $i$, then it stays in $[i]$ forever, visiting each of these recurrent states infinitely many times.
• If MC begins in transient i, then

(a) if # of transient states is finite, it will eventually hit a recurrent state (say, j) and then stay in [j] forever, as above.

(b) if # of transient states is infinite, then might have
(i) Same behavior as (a) OR
(ii) travel through set of transient states forever.

\[ \sum_{n=1}^{\infty} p^n_i, i = 8 \]

\( i \) is recurrent
**PROGRESS SO FAR:** $f_{ij}$

<table>
<thead>
<tr>
<th>$i/j$</th>
<th>trans.</th>
<th>$??$</th>
<th>$0$</th>
<th>recur.</th>
</tr>
</thead>
<tbody>
<tr>
<td>trans.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>recur.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Know $f_{ij}$ only depends on $[i,j]$. 
WLOG in Computing?

we may take $j$ to be absorbing: $P_{ij} = 1$.

We'll compute $f_{ij}$ (for?) by conditioning on $X_i$:

$[i \text{ transient, } j \text{ absorbing}]$

$[\text{varying} \quad \text{fixed}]$

$f_{ij} = P_{ij} + \sum_{k \text{ transient}} P_{ik} f_{kj}$
Theorem. If the number of transient states is finite and \( j \) is an absorbing state, then the system

\[
\phi_i = \sum_{j \in \text{transient}} P_{ij} \phi_j + \sum_{k \in \text{transient}} \sum_{i \in \text{transient}} P_{ik} \phi_k
\]

has the unique solution

\[
\phi_i = f_i \phi_j
\]
Pf. Only need to show uniqueness of sol’n.
We’ll use method of successive substitutions:

\[ \phi_i = P_{ij} + \sum_{k \neq i} P_{ik} \phi_k \]

\[ = P_{ij} + \sum_{k \neq i} P_{ik} \left( P_{kj} + \sum_{l \neq k} P_{kl} \phi_l \right) \]

\[ = P_{ij} + f_{ij} + \sum_{l \neq i} P_{il} \phi_l \]
\[ \phi_i = f_{i,j} + \sum_{\text{let } j}^\leq n \sum_{\text{let } k} p_{i,k} \phi_k \]

Where \( f_{i,j} := p_{i,k} \text{ if } j \leq n \).

Repeat:

\[ \phi_i = f_{i,j} + \sum_{\text{let } k}^\leq 3 \sum_{\text{let } k}^3 p_{i,k} \phi_k \]

\[ \vdots \]

\[ \forall n: \phi_i = f_{i,j} + \sum_{\text{let } k}^n \sum_{\text{let } k}^n p_{i,k} \phi_k \]

Now let \( n \to \infty \).
\[ \phi_i = f_{ij} + \sum_{k \in T} Z \cdot \phi_k \]

\[ = f_{ij} \]

General cond. under which closed form expression for \( f_{ij} \) exists: the MC \( X \) is also a martingale.
MARTINGALES (mgale)

Idea: A process is a mgale if it "stays put -- on the average".

DEFN: A process $X = (X_n)_{n \geq 0}$ with

$$E(X_{n+1} | X_0, \ldots, X_n) = X_n$$
is called a martingale. on $S_0, \ldots, S_T$

**Note.** A MC is also a martingale iff

$$\sum_{j \neq i} P_{ij} = 1, \forall i.$$ 

**Observation.** If states $1, \ldots, d-1$ are all non-absorbing, then
they are all transient and states 0 & d are absorbing

We know that \( \phi_{id} \) is the unique sol'n to

\[
\phi_i = \text{P}_{id} + \sum_{k=1}^{d-1} \text{P}_{ik} \phi_k
\]

Notice that

\[
\phi_i \equiv \frac{i}{d}
\]

is a sol'n.

(Check this.)
So \[ \text{fid} = \frac{i}{d} \].

The gambler's ruin chain is an example of a mgale MC, so

\[ \text{fid} = \frac{i}{d}, \]

\[ \text{fo} = \frac{d - i}{d}. \]