LIMIT THEOREMS FOR RENEWAL PROCESSES

PROP. 3.3.1 (SLLN for renewal processes)

Let $\mu = \text{EX}_1$. Then

$$\frac{N(t)}{t} \xrightarrow{\text{w.p.1}} \frac{1}{\mu}$$

as $t \to \infty$.

The diagram shows the points $0$, $S_{N(t)}$, and $S_{N(t)+1}$.
Proof.

\[
\frac{S_N(t)}{N(t)} \leq \frac{t}{N(t)} \leq \frac{N(t) + 1}{N(t)} \leq \frac{N(t)}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)+1}
\]

\[\text{SLLN for } S_t \text{ as } t \to \infty\]

\[\mu \quad \text{wp}\]

Therefore

\[
\frac{t}{N(t)} \to \mu \quad \text{wp}
\]

i.e.

\[
\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{wp}
\]

\[N(t) \to \infty \quad \text{wp}\]

because

\[P(\bigcap_{n=1}^{\infty} \{X_n = \infty\}) \leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0.\]
CLT for $N$:

**Thm. 3.3.5**

**Derivation.**

If we're to have a CLT for $N$, we need to find how $n$ should relate to $t$, so that

$$P\{N(t) < n\} \approx \Phi(z)$$

Indeed,

$$P\{N(t) < n\} = P\{S_n > t\}$$

will be

$$= 1 - \Phi(z) = \Phi(-z)$$

**Switching Relation**
if and only if

$$t = \mu n + z \sigma \sqrt{n}$$

where \( \mu := \mathbb{E}X \),
\( \sigma^2 := \mathbb{V}X \),

So \( P\{N(t) < n\} \) will be \( \equiv \chi(2) \)

iff

$$t = \mu n - z \sigma \sqrt{n}$$

iff

$$\mu (\sigma^2 - z \sigma \sqrt{n}) - t = 0$$

iff

$$n = \frac{2 \sigma \pm \sqrt{2 \sigma^2 + 4 \mu t}}{2 \mu}$$

$$= \frac{2 \sigma}{2 \mu} \pm \sqrt{\frac{2 \sigma^2}{2 \mu} + \frac{4 \mu t}{2 \mu}}$$
iff

\[ n = \frac{2\sigma}{2\mu^2} + \frac{t}{\mu} \pm \frac{20}{\mu} \sqrt{\frac{t}{\mu} + \frac{\sigma^2 \rho^2}{4\mu^2}} \]

\[ = \frac{t}{\mu} \pm \frac{20}{\mu} \sqrt{\frac{t}{\mu}} \]

\[ = \frac{t}{\mu} + 20 \sqrt{\frac{t}{\mu}}. \]

This suggests

\[ \Phi(z) = \Pr \{ N(t) < \frac{t}{\mu} + 20 \sqrt{\frac{t}{\mu}} \} \]

\[ = \Pr \left\{ \frac{N(t) - \frac{t}{\mu}}{\sigma \sqrt{\mu}} < z \right\}. \]

This is correct!
NEXT:
THM. 3.3.4

Elementary Renewal Thm:

\[
\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty
\]

\[
\text{with} \quad \frac{1}{\mu} \quad \text{if} \quad \mu < \infty
\]

\[
\frac{N(t)}{t} \to \frac{1}{\mu}
\]

does not immediately give

\[ \text{E} \cdot \frac{1}{\mu} \]
DEFN. [not most general]

An integer-valued r.v. \( N \) is said to be a stopping time for an independent sequence of r.v.'s \( X_1, X_2, \ldots \) if

\[ A_n: \{ N = n \} \text{ and } (X_{n+1}, X_{n+2}, \ldots) \]

are independent.
EXAMPLE. Consider a renewal process. Let

\[ X_n^i = n^{th} \text{ interarrival time} \]

and consider (for fixed \( t \))

\[ N(t) + 1 \]

CLAIM. \( N(t)^+ \) is a stopping time.

Proof. \( \{ N(t)^+ = n \} \)

\[ = \{ N(t) = n-1 \} \]

\[ = \{ S_{n-1} \leq t < S_n \} \]

is independent of \( (X_{n+1}, X_{n+2}, \ldots) \).
EXAMPLE. Same set-up.

CLAIM. \( N(t) \) is \( \textbf{NOT} \) a stopping time.

Why not? \[ \{ N(t) = n \} = \{ S_n \leq t < S_{n+1} \}. \]

WALD'S EQN. (THM. 3.3.2)

If \( X_1, X_2, \ldots \) are i.i.d. r.v.'s having finite expectation \( \mu \),
and if \( N \) is a stopping time for \( X_1, X_2, \ldots \), and if \( E[N] < \infty \),
\[ z \approx \sum_{n=1}^{\infty} \frac{\mu \cdot \beta \cdot \mathbb{P}(N \geq n)}{n^2 \mathbb{P}(N \geq n)} \]

\[ \approx \mu \cdot \beta \cdot \mathbb{E}N \cdot \mathbb{E}_n \]

X has a i.i.d. with mean

\[ = \mu \cdot \mathbb{E}N, \]
ERT,

\[
\frac{m(t)}{t} \xrightarrow{t \to \infty} \frac{1}{\mu} \quad \text{(where } \frac{1}{\infty} = 0).\]

Proof. First suppose \( \mu < \infty \).

Then \( S_{N(t)+1} > t \)

\( \Rightarrow E[S_{N(t)+1}] = \mathbb{E}[m(t)+1] \)

\( > t \)

\( \Rightarrow \frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t} \)

\( \Rightarrow \liminf_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu} , \)
For an upper bound, we would like to have an inequality of the form $S \leq t$.

Fix $M$. Define a new renewal process $\mathcal{N}$ in terms of

$$X_n := \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases}$$

$$S_n := \frac{\mathcal{N}}{\mathcal{N}} X_i, \quad \overline{N}(t) = \sup \{ n : S_n \leq t \}$$

Note that $S_n \leq S_n$

$$\Rightarrow \overline{N}(t) \geq N(t)$$

$$\Rightarrow \frac{\overline{N}(t)}{m(t)} \geq m(t).$$
Observe
\[ \overline{S}_{N(t)+1} = \overline{S}_{N(t)} + X_{N(t)+1} \]
\[ \leq t + M \]
\[ \Rightarrow [\overline{m}(t)+1] \mu_m \leq t + M \]

where \( \mu_m := E X \)
\[ \Rightarrow \limsup_{t \to \infty} \frac{\overline{m}(t)}{t} \leq \limsup_{t \to \infty} \left[ \frac{t + M}{t} \right] \]
\[ = \frac{1}{\mu_m} \]
$$\limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu} \forall \mu$$

As $M \uparrow \infty$, MCT implies that $\mu_M \uparrow \mu$,

so

$$\limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{1}{\mu}.$$ 

If $\mu = \infty$, then still holds

because Wald was applied only

to truncated $X$'s to get $\mu_M$, and

MCT still works to get

$$\lim_{t \to \infty} \frac{m(t)}{t} \leq 0.$$
DEFN. A r.v. $X \in \mathbb{R}$ is said to have an arithmetic distribution if there exists $d > 0$ s.t.

$$\forall n \in \mathbb{Z}, \quad \mathbb{P}(X = nd) = 1.$$
Fact. The span of an integer-valued r.v. having a geometric distribution equals non-degenerate.

\[ \gcd \{ n \in \mathbb{Z} : P[X = n] > 0 \} . \]

Example.

\[ 2 = \gcd(4, 6) = \text{span of dist'phenomenon} \]
Suppose
\[ P[X = e] = \frac{1}{2} \]
\[ P[X = \pi] = \frac{1}{2} \].

Q: arithmetic or not?

A: If arithmetic, let \( d \) be span of dist'

Then
\[ e = nd \] for some \( n \)
\[ \pi = md \] for some \( m \)

\[ d = \frac{e}{n} = \frac{\pi}{m} \Rightarrow \frac{e}{\pi} = \frac{n}{m} \]
\( d \) is rational.