Today:
I begin by discussing how to simulate a non-stationary Poisson process, based on ideas on p. 80 in the text.

**NEXT: Compound Poisson Processes**
DEFIN. A stochastic proc.

\[ X = (X(t); t \geq 0) \]

is called a **compound Poisson process** if it can be represented in the form

\[ \forall t \geq 0: \quad X(t) = \sum_{i=1}^{N(t)} X_i \]

where \( X_1, X_2, \ldots \) are i.i.d. r.v.'s, independent of

\[ N = \text{Poi process w/ rate } \lambda. \]
If \( W = \sum_{i=1}^{N} X_i \) (*)

is a compd Pois r.v.

\( (\text{def}) \) is its representation

\( (X_1, X_2, \ldots \text{are indep.}) \)

\( N \sim \text{Poi}(\lambda) \)

Then we say

\( W \sim \text{compd Pois}(\lambda, F) \)
EXAMPLE 2.5(A), p. 83

Poisson proc $N$ w/ rate $\alpha$

An event occurring at times will, indp. of past, result in a contrib. whose value is a r.v. with dist $F_\lambda$.

Let $W := \sum_{i=1}^{N(t)} X_i$  

Claim: $W$ has a compound Poisson dist w/ parameters $(\alpha t, F)$ where $F(x) = \frac{1}{t} \int_{s=0}^{t} F_\lambda (x) ds$
Let's assume that each of the dist's $F_s$ has a continuous density $f_s$.

$$F_w(w) = P\{W \leq w\}$$

[This really should be made precise using $\lim_{h \to 0} P\{W \in [w, w+h]\}$]

$$= \sum_{n=0}^{\infty} P\{N(t) = n\} P\{W \leq w | N(t) = n\}$$

But

$$P\{N(t) = n\} = e^{-\lambda t} \left(\frac{\lambda t}{n!}\right)^n$$
\[ P\{W = w \mid N(t) = n\} = \int_{a_1 < s_2 < \ldots < s_n < t} f_{s_1, \ldots, s_n}(a_1, \ldots, a_n) \times P\{W = w \mid N(t) = n, S_1 = a_1, \ldots, S_n = a_n\} \, da_1 \cdots da_n \]

Now
\[ P\{W = w \mid N(t) = n, S_1 = a_1, \ldots, S_n = a_n\} = \mathbb{P}\{\bigoplus_{i=1}^{n} Y_{S_i} = w\} \]

where \( Y_{S_i} \sim F_{S_i} \).
Then
\[ P\{W = w\} = \sum_{n=0}^{\infty} \frac{\left(\frac{e^{-at}}{n!}\right)^n}{n!} \]

\[ \times \int \cdots \int f_{S_1, \ldots, S_n}(s_1, \ldots, s_n) \]
\[ 0 < s_1 < s_2 < \cdots < s_n < t \]

\[ \times P\{\bigcap_{i=1}^{n} Y_i = w\} ds_1 \cdots ds_n \]

**KEY STEP:**

\[ \sum_{n=0}^{\infty} \frac{\left(\frac{e^{-at}}{n!}\right)^n}{n!} = \prod_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{e^{-at}} \cdot \frac{1}{(at)^n} \]

where \( U_1, \ldots, U_n \sim \text{i.i.d. unif}(0,t) \)

and, conditionally given \( U_1 = u_1, \ldots, U_n = u_n \), \( Y_1, \ldots, Y_n \) are indep. with each \( Y_i \sim F_v \).
\[ \text{and U's are indep. of Y's.} \]
\[ \text{and U's \& Y's are indep. of N.} \]

If we can show that the r.v.'s
\[ Z_1, \ldots, Z_n, \]

when \( Z_i := Y_{u_i} \),

are indep., then
\[ P\{W = w\} = \sum_{n=0}^\infty \frac{e^{-At} (At)^n}{n!} \]
\[ \times P\{ \sum_{i=1}^n Z_i = w\} \]

\[ = P\{N(w) \leq 1\} \]

\[ \forall i : P\{Z_i \leq z\} = P\{Y_{u_i} \leq z\} \]
\[= \frac{1}{t} \int_{u=0}^{t} P(Y_u \leq z) \, du \]
\[= \frac{1}{t} \int_{u=0}^{t} F_u(z) \, du = F(z).\]

What's left:

Show that $Z_1, \ldots, Z_n$ are indep.

\[P\{Z_1 \leq z_1, \ldots, Z_n \leq z_n\} = P\{Y_{u_1} \leq z_1, \ldots, Y_{u_n} \leq z_n\}\]
\[= \frac{1}{t^n} \int_{u_1=0}^{t} \cdots \int_{u_n=0}^{t} P(Y_{u_1} \leq z_1, \ldots, Y_{u_n} \leq z_n) \, du_1 \cdots du_n\]
\[= \frac{1}{t^n} \int_{u_1=0}^{t} \cdots \int_{u_n=0}^{t} F_{u_1}(z_1) \cdots F_{u_n}(z_n) \, du_1 \cdots du_n\]
\[= \prod_{i=1}^{n} \left[ \frac{1}{t} \int_{u_i=0}^{t} F_{u_i}(z_i) \, du_i \right].\]
Poisson Process

In the Plane

"Defn."

The process

\[ N = \{ N(A) : A \text{ is a region in the plane} \} \]

is called a Poisson process in \( A \)

if \( N \) is non-negative integer-valued and

\[ N(A) \sim \text{Poi} \left( \frac{\lambda}{\text{area of } A} \right) \]
\[ A_1 \cap A_2 \cap \cdots \cap A_n \]

\[ N(A_1), \ldots, N(A_n) \] are independent r.v.'s.

**How to simulate?**

**Theorem.** For any region \( A \),

\[ N(A) = n, \]

the \( n \) points of the Poisson process are distributed as \( n \) i.i.d. uniform pts. from \( A \).
X - Poisson proc. w/ rate \( \lambda \)

\[
\begin{array}{c}
0\quad x\quad x\quad x\quad x\quad x\quad x\quad x\quad x\quad f
\
\end{array}
\]

Color red (Type 1) w/ \( P(s) \)

Color green (Type 2) w/ \( 1 - P(s) \)

\[
\frac{1}{t} \int_{s=0}^{t} P(s) \, ds
\]

For a 2-dim nonhomogeneous Poisson process, \( \text{intensity fun.} \)

\[
N(A) \sim \text{Poi} \left( \int_A \lambda(x) \, dx \right)
\]
Problem 2.10,

(a) By conditioning on the time of arrival of the first bus,

\[ ET = \int_{x=0}^{s} \lambda e^{-\lambda x} (x+R) dx + e^{-\lambda s} (s+W) \]

\[ = \frac{1}{x} + R + e^{-\lambda s} \left[ W - \left( \frac{1}{x} + R \right) \right] \]

(b) For the optimal \( s \),

\[ ET = \min(W, \frac{1}{x} + R) \]
HW #4
Due Wed. 2/21

Problems 2.11
  2.30
  2.33
  2.35
  2.37
  2.39

In Chap. 3, we'll cover §§ 3.1-3.3 in detail & skim rest.