Homework No.6, 550.696, Due April 28, 2014.

1. Kraichnan in 1976 considered a model of "vortex blobs" with initial stream function of the form

$$\psi_0(\mathbf{x}) = \frac{\omega_0}{k_0^2} e^{-r^2/2D^2} \cos(\mathbf{k}_0 \cdot \mathbf{x})$$

in an external strain field corresponding to a large-scale velocity

$$U_1(\mathbf{x}, t) = a(t)x_1, \quad U_2(\mathbf{x}, t) = -a(t)x_2, \quad a(t) > 0.$$

(a) Show that the initial velocity and vorticity of the blob are given by

$$\mathbf{u}_0(\mathbf{x}) = \frac{\mathbf{k}_0^\perp}{k_0^2} \omega_0 e^{-r^2/2D^2} \sin(\mathbf{k}_0 \cdot \mathbf{x}), \quad \omega_0(\mathbf{x}) = \omega_0 e^{-r^2/2D^2} \cos(\mathbf{k}_0 \cdot \mathbf{x})$$

up to corrections of order $O((k_0 D)^{-1})$ for $k_0 D \gg 1$, and likewise the energy and enstrophy are given to leading order by

$$E_0 = \frac{1}{k_0^2}\Omega_0, \quad \Omega_0 = \frac{\pi \omega_0^2 D^2}{4}.$$

Hint: You may find useful the Bessel-function integral $\int_0^\infty du\ e^{-u} J_0(2\sqrt{\alpha u}) = e^{-\alpha}$. E.g. see A. Erdélyi, *Higher Transcendental Functions*, vol. I, formula 6.10.8.

(b) Show that under passive distortion $(\partial_t + \mathbf{U} \cdot \boldsymbol{\nabla})\omega = 0$ by the large-scale field, vorticity evolves into $\omega(\mathbf{x}, t) = \omega_0 \exp\left[-(e^{-2\beta(t)}x_1^2 + e^{2\beta(t)}x_2^2)/2D^2\right] \cos(\mathbf{k}(t) \cdot \mathbf{x})$, an elongated elliptical-shaped packet, and the velocity likewise evolves into

$$\mathbf{u}(\mathbf{x}, t) = \frac{\mathbf{k}^\perp(t)}{k^2(t)} \omega_0 \exp\left[-(e^{-2\beta(t)}x_1^2 + e^{2\beta(t)}x_2^2)/2D^2\right] \sin(\mathbf{k}(t) \cdot \mathbf{x}),$$

with $\mathbf{k}(t) = (e^{-\beta(t)}k_1^{(0)}, e^{\beta(t)}k_2^{(0)})$ and $\beta(t) = \int_0^t a(s)ds$, and thus the energy and enstrophy evolve into

$$E(t) = \frac{1}{k^2(t)}\Omega_0, \quad \Omega(t) = \Omega_0.$$

(c) Now assume that there is a random ensemble of vortex blobs with an isotropic distribution of wavenumbers $\mathbf{k}$ of magnitude k_0. Show that $\langle k^2(t) \rangle = \cosh[2\beta(t)]k_0^2$, but that

$$\langle E(t) \rangle = \langle k^{-2}(t) \rangle \Omega_0 = k_0^{-2}\Omega_0 = E_0,$$

so that mean energy of the small-scale blobs is conserved! Why doesn't this result contradict the "vortex-thinning mechanism" of 2D inverse energy cascade?

2. This problem concerns the quantity for an incompressible 2D velocity field $\mathbf{u}$ defined by

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma}_\ell(\mathbf{u},\mathbf{u}) &= \frac{1}{\ell^2}\int d^2r\ (\partial_i\partial_j^\perp G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\delta u_j(\mathbf{r}) \\ &\quad -\frac{1}{\ell^2}\int d^2r\ G_\ell(\mathbf{r})\delta u_i(\mathbf{r})\cdot\int d^2r\ (\partial_i\partial_j^\perp G)_\ell(\mathbf{r})\delta u_j(\mathbf{r}) \\ &\quad -\frac{1}{\ell^2}\int d^2r\ (\partial_j^\perp G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\cdot\int d^2r\ (\partial_i G)_\ell(\mathbf{r})\delta u_j(\mathbf{r})\end{aligned}$$

that appears in the DiPerna-Lions theory applied to 2D Euler. Show that if $\mathbf{u}$ is a smooth field, then the first term on the right converges to $-\epsilon_{jk}(\mathbf{D}^2)_{jk}$, the second term to 0, and the third to $\epsilon_{jk}(\mathbf{D}^2)_{jk}$, with $D_{ij} = \partial u_i/\partial x_j$. Thus, the net limit is zero. In fact, using Homework #5, Problem 3 (a), show that $\epsilon_{jk}(\mathbf{D}^2)_{jk} = 0$.

3. Derive the following typical estimate used in the DiPerna-Lions theory:

$$\|h(\bar{\omega}_\ell) - h(\omega)\|_1 \leq 2^{(p-1)/p}\max\left\{M_h, 2^{p-1}C\|\omega\|_p^{p-1}\right\}\cdot\|\bar{\omega}_\ell - \omega\|_p$$

for

$$h \in \mathcal{H}_p = \{h:\ h \in C^1(\mathbb{R})\ \ \&\ \ |h'(z)| \leq C|z|^{p-1} \text{ for } |z| \geq R\}$$

where $M_h = \max_{|z|\leq R}|h'(z)|$. Here we define as usual $\|f\|_p = \left[\frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} d^dx\ |f(\mathbf{x})|^p\right]^{1/p}$.

Hint: Use the mean-value theorem to write

$$h(\bar{\omega}_\ell(\mathbf{x})) - h(\omega(\mathbf{x})) = h'(\omega_\ell^\theta(\mathbf{x}))\cdot(\bar{\omega}_\ell(\mathbf{x}) - \omega(\mathbf{x}))$$

where $\omega_\ell^\theta(\mathbf{x}) = \theta(\mathbf{x})\bar{\omega}_\ell(\mathbf{x}) + (1-\theta(\mathbf{x}))\omega(\mathbf{x})$. Use Young's inequality for convolutions to show that $\|\omega_\ell^\theta\|_p \leq 2\|\omega\|_p$ and use the definition of $\mathcal{H}_p$ to show that

$$\|h'(\omega)\|_{p/(p-1)} \leq 2^{(p-1)/p}\max\left\{M_h, C\|\omega\|_p^{p-1}\right\}.$$

From the above estimates conclude that

$$\lim_{\ell\to 0}\|h(\bar{\omega}_\ell) - h(\omega)\|_1 = 0.$$

Remark: For the proof that $\lim_{\ell\to 0}\|\bar{\omega}_\ell - \omega\|_p = 0$, see E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Chapter III, Section 2.2, Theorem 2.