Homework No.5, 550.696, Due April 14, 2014.

1. (a) Use the results of Homework #4, Problem 3, to calculate the "nonlinear model", i.e. the contributions only from resolved scale velocity $\bar{\mathbf{u}}_\ell$ and to 1st-order in gradients, for the subscale vorticity transport vector using the formula

$$\boldsymbol{\sigma}_\ell^\perp = \boldsymbol{\nabla}\cdot\tau_\ell(\mathbf{u},\mathbf{u}) - \boldsymbol{\nabla}\left(\frac{1}{2}\tau_\ell(u_k,u_k)\right),$$

The approximation exploits the UV scale-locality of the above expression for $\boldsymbol{\sigma}_\ell$.

(b) Show that the result is identical to that obtained from $\boldsymbol{\sigma}_\ell = \tau_\ell(\omega,\mathbf{u})$, i.e.

$$\sigma_{\ell i}^{NL} = \frac{1}{2}C\ell^2\bar{D}_{\ell ij}\frac{\partial\bar{\omega}_\ell}{\partial x_j},$$

although the UV scale-locality of this formula for $\boldsymbol{\sigma}_\ell$ cannot be directly established.

2. Consider a KB energy spectrum modified with a logarithm, for $0 < p < 1$:

$$E(k) = C\eta^{2/3}k^{-3}\ln^{-p}(k/k_f), \quad k_f < k < k_{uv},$$

and zero otherwise.

(a) Show that the rms magnitude of the coarse-grained strain at scales $> \ell$ with this spectrum is given for $k_f < 2\pi/\ell < k_{uv}$ by

$$(\bar{S}_\ell)_{rms} \sim C'\eta^{1/3}\ln^{\frac{1-p}{2}}(2\pi/k_f\ell).$$

Employ a sharp spectral filter at wavenumber $k = 2\pi/\ell$, for simplicity.

(b) Show likewise that the rms magnitude of the coarse-grained vorticity-gradient at scales $> l$ is given for $k_f \ll 2\pi/\ell < k_{uv}$ by

$$(\nabla\bar{\omega}_\ell)_{rms} \sim C''\eta^{1/3}\frac{1}{\ell\ln^{\frac{p}{2}}(2\pi/k_f\ell)}.$$

(c) Show then that the rough estimate of the enstrophy flux

$$Z_\ell \sim (const.)\ell^2(\bar{S}_\ell)_{rms}(\nabla\bar{\omega}_\ell)^2_{rms}$$

becomes independent of ℓ only for $p = 1/3$.

3. (a) Show that for an incompressible velocity field $\mathbf{u}$ in 2D that $D_{ij} = \partial u_i/\partial x_j$ satisfies

$$\mathrm{tr}(\mathbf{D}^2) = -2 \det(\mathbf{D}).$$

(b) Use part (a) to show that if $\frac{1}{2}\omega^2 > S^2 = S_{ij}S_{ij}$, then $\mathbf{D}$ has a conjugate pair of pure imaginary eigenvalues and if $\frac{1}{2}\omega^2 < S^2$, a pair of real eigenvalues of equal magnitudes and opposite signs.

(c) If $\mathbf{u}$ solves 2D Navier-Stokes equation, how is $\mathrm{tr}(\mathbf{D}^2)$ related to the pressure p?

4. (a) If $\mathbf{S}$ is any symmetric, traceless 2×2 matrix, show that it may be written in the "polar form"

$$\mathbf{S} = \sigma \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

where $\sigma = \sqrt{S_{11}^2 + S_{12}^2} = \sqrt{S_{21}^2 + S_{22}^2}$, and that

$$\mathbf{e}_+ = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad \mathbf{e}_- = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

are the eigenvectors with eigenvalues $+\sigma, -\sigma$, respectively.

(b) If $\mathbf{S}_1, \mathbf{S}_2$ are both symmetric, traceless 2×2 matrices written as in (a), show that their matrix scalar product is

$$\mathbf{S}_1 : \mathbf{S}_2 = 2\sigma_1\sigma_2 \cos(2(\theta_1 - \theta_2)).$$

(c) Show that if $\boldsymbol{\epsilon}$ is the 2D Levi-Civita tensor, or in matrix form

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and if $\mathbf{S}$ is a symmetric, traceless 2×2 matrix, then

$$\mathbf{S}\boldsymbol{\epsilon} = -\boldsymbol{\epsilon}\mathbf{S}.$$

(d) Defining the *skew* of $\mathbf{S}$, or $\tilde{\mathbf{S}}$, by either of the above expressions, show that $\tilde{\mathbf{S}} : \mathbf{S} = 0$ and, in fact, that its eigenframe is rotated by $+45°$ to the frame of $\mathbf{S}$.