Homework #4 – Solutions

Problem 1. As in Homework #1, Problem 2(a)

$$\hat{\bar{u}}_\ell(\mathbf{k},t) = \hat{G}(-\ell\mathbf{k})\,\hat{u}(\mathbf{k},t)$$

where $\hat{u}(\mathbf{k},t) = \frac{1}{|D|}\int_D d^2x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, u(\mathbf{x},t)$ and the discrete convolution theorem was employed. Parseval's equality implies

$$E_{>\ell}(t) = \frac{1}{2}\sum_{\mathbf{k}} |\hat{G}(-\ell\mathbf{k})|^2\, |\hat{u}(\mathbf{k},t)|^2.$$

Using the definition of the energy spectrum

$$E(k,t) = \frac{1}{2}\sum_{\mathbf{k}'} \delta(k-|\mathbf{k}'|)\,|\hat{u}(\mathbf{k}',t)|^2,$$

it follows that

$$E_{>\ell}(t) = \int_0^\infty dk\, |\hat{G}(\ell k)|^2\, E(k,t)$$

when $\hat{G}(\mathbf{k})$ depends only upon $k=|\mathbf{k}|$. Now calculate the derivative $(d/dt)_{NL}$ of both sides, where $\left(\frac{d}{dt}\right)_{NL}$ denotes the time-derivative only under the nonlinear terms in the dynamical equations.

Since

$$\Pi_\ell(t) = \frac{1}{|D|}\int_D d^2x\, \Pi_\ell(\mathbf{x},t) = -\left(\frac{d}{dt}\right)_{NL} E_{>\ell}(t)$$

and

$$T(k,t) = \left(\frac{d}{dt}\right)_{NL} E(k,t),$$

it follows that

$$\Pi_\ell(t) = -\int_0^\infty dk \, |\hat{G}(\ell k)|^2 \, T(k,t).$$

The definition of the spectral flux $\Pi(k,t) = -\int_0^k dk' \, T(k',t)$ also implies that

$$T(k,t) = -\frac{d}{dk} \Pi(k,t)$$

so that

$$\Pi_\ell(t) = \int_0^\infty dk \, |\hat{G}(\ell k)|^2 \frac{d}{dk} \Pi(k,t).$$

Integration by parts using $\hat{G}(\infty) = 0$ and $\Pi(0,t) = 0$ gives

$$\Pi_\ell(t) = -\int_0^\infty dk \, \frac{d}{dk} |\hat{G}(\ell k)|^2 \, \Pi(k,t)$$

$$= \int_0^\infty dk \, P_\ell(k) \Pi(k,t)$$

with

$$P_\ell(k) \equiv \frac{d}{dk} |\hat{G}(\ell k)|^2.$$

(b) By the fundamental theorem of calculus

$$\int_0^\infty dk\, P_\ell(k) = -\int_0^\infty dk\, \frac{d}{dk}|\hat{G}(\ell k)|^2$$

$$= |\hat{G}(0)|^2 - |\hat{G}(\infty)|^2$$

$$= 1 - 0 = 1.$$

Also,

$$P_\ell(k) = -\frac{d}{dk}|\hat{G}(\ell k)|^2$$

$$= -\ell\left(\frac{d}{du}|\hat{G}(u)|^2\right)_{u=k\ell} = \ell\, P(\ell k).$$

Finally, a function which is non-increasing is differentiable almost everywhere with respect to Lebesgue measure.

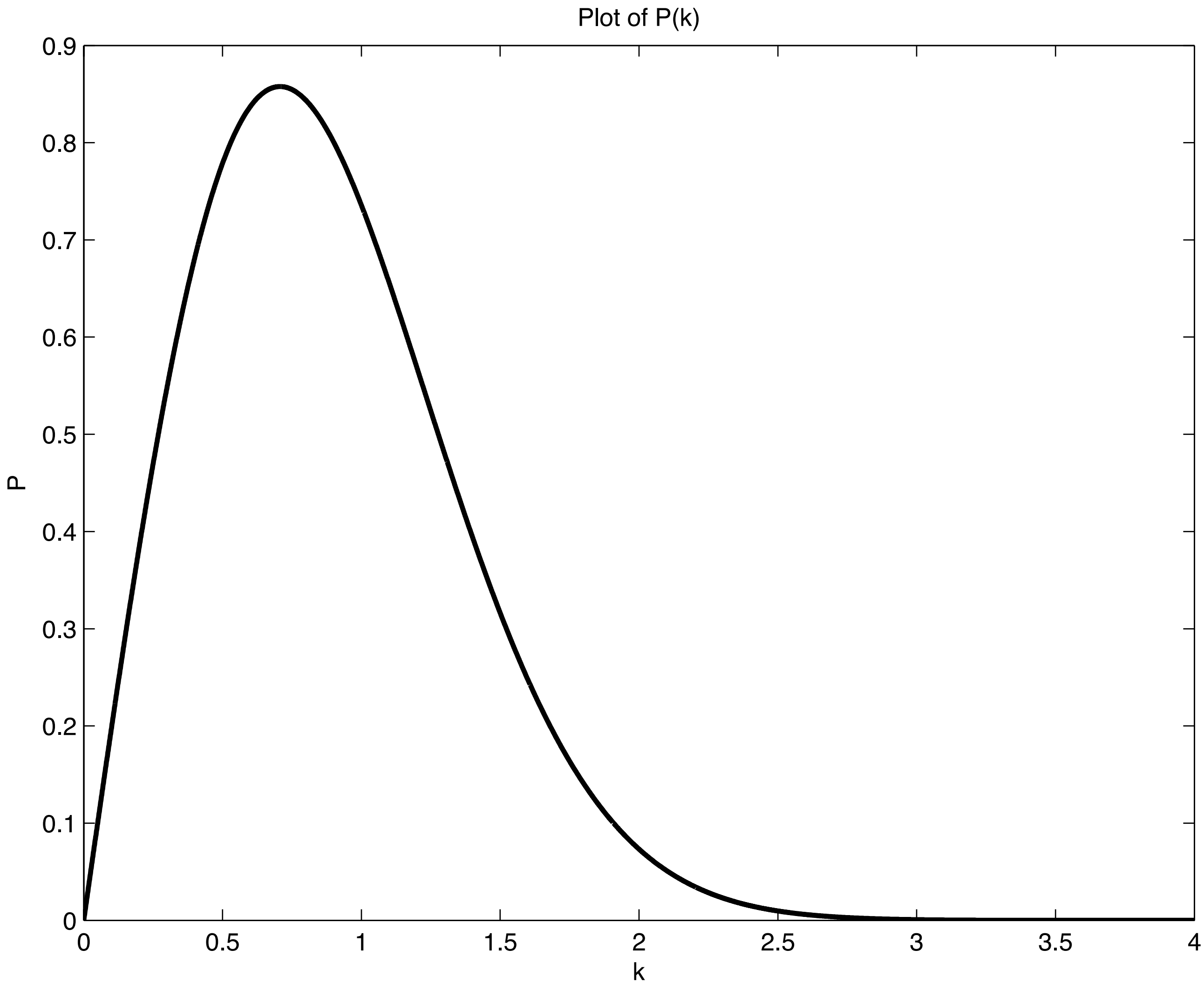
If $|\hat{G}(k)|^2$ is non-increasing then

$$P(k) = -\frac{d}{dk}|\hat{G}(k)|^2$$

$$\geq 0.$$

Finally, a plot of $P(k) = -\frac{d}{dk}|e^{-k^2/2}|^2 = -\frac{d}{dk}e^{-k^2} = 2k e^{-k^2}$

for the Gaussian case appears on the following page.



Plot of P(k)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
P
0
0.5
1
1.5
2
2.5
3
3.5
4
k

Problem 2. (a) First consider the 2D Euler dynamics. Direct time-differentiation gives

$$\partial_t(\omega\omega') = -(u\cdot\nabla)\omega\cdot\omega' - (u'\cdot\nabla')\omega'\cdot\omega$$

$$= -(u\cdot\nabla)\omega\cdot\omega' - (u'\cdot\nabla)\omega'\cdot\omega$$

$$= \underbrace{-(u\cdot\nabla)\omega\cdot\omega' - (u\cdot\nabla)\omega'\cdot\omega}_{-(u\cdot\nabla)(\omega\omega')} - (\delta u\cdot\nabla)\omega'\cdot\omega.$$

In the last term use $\omega = \omega' - \delta\omega$ to write

$$-(\delta u\cdot\nabla)\omega'\cdot\omega = \underbrace{-(\delta u\cdot\nabla)\omega'\cdot\omega'}_{-(\delta u\cdot\nabla)\left(\frac{1}{2}|\omega'|^2\right).} + (\delta u\cdot\nabla)\omega'(\delta\omega)$$

Using incompressibility, we can now rearrange the above as

$$\partial_t\left(\tfrac{1}{2}\omega\omega'\right) + \nabla_x\cdot\left[\left(\tfrac{1}{2}\omega\omega'\right)u + \tfrac{1}{4}|\omega'|^2\delta u\right] = \tfrac{1}{2}(\delta u\cdot\nabla)\omega'(\delta\omega).$$

However,

$$\nabla_r\cdot\left[\delta u(r)|\delta\omega(r)|^2\right] = \delta u(r)\cdot\nabla_r\left[|\delta\omega(r)|^2\right] \quad \text{by incompressibility}$$

$$= 2\,\delta u(r)\cdot\nabla_r\omega'\,\delta\omega(r) \quad \text{by product rule}$$

and $\nabla_r\delta\omega(r) = \nabla_r(\omega'-\omega) = \nabla_r\omega' = \nabla_x\omega'$. Hence,

$$\tfrac{1}{2}(\delta u\cdot\nabla)\omega'(\delta\omega) = \tfrac{1}{4}\nabla_r\cdot\left[\delta u(r)|\delta\omega(r)|^2\right].$$

The linear term in the dynamics is trivial, since by product rule

$$\partial_t \left(\tfrac{1}{2} w w' \right) = \tfrac{1}{2}(-\alpha w) w' + \tfrac{1}{2} w (-\alpha w') = -\alpha w w'.$$

The viscous dynamics gives

$$\begin{aligned}
\partial_t \left(\tfrac{1}{2} w w' \right) &= \tfrac{1}{2} \nu \Delta w \cdot w' + \tfrac{1}{2} w \cdot \nu \Delta w' \\
&= \nabla \cdot \left[\tfrac{1}{2} \nu \nabla w \cdot w' + \tfrac{1}{2} w \cdot \nu \nabla w' \right] \\
&\qquad - \nu \tfrac{1}{2} \left[\nabla w \cdot \nabla w' + \nabla w \cdot \nabla w' \right] \\
&= \nabla \cdot \left[\nu \nabla \left(\tfrac{1}{2} w w' \right) \right] - \nu \nabla w \cdot \nabla w'.
\end{aligned}$$

Putting all of these separate cases together gives the desired result. Finally, for $\nu = 0$, all of the derivatives with respect to $\mathbf{x}$, t and $\mathbf{r}$ can be moved to a test function $\Phi(\mathbf{x}, t, \mathbf{r})$ and the result is finite as long as all of the following functions are integrable (i.e. L^1):

$$w w', \quad w w' \mathbf{u}, \quad (w')^2 \delta \mathbf{u}, \quad \delta \mathbf{u} |\delta w|^2 \in L^1$$

as functions of $(\mathbf{x}, t, \mathbf{r}) \in \mathbb{T}^2 \times [0, T] \times \mathbb{T}^2$.

(b) When smeared with respect to $\mathbf{r}$, the equation in part (a) becomes

$$(*)\quad \partial_t\left(\tfrac{1}{2}\omega\bar{\omega}_\ell\right) + \nabla_x\cdot\left[\left(\tfrac{1}{2}\omega\bar{\omega}_\ell\right)\mathbf{u} + \tfrac{1}{4}\overline{(\omega^2\mathbf{u})}_\ell - \tfrac{1}{4}\overline{(\omega^2)}_\ell\mathbf{u} - \nu\nabla_x\left(\tfrac{1}{2}\omega\bar{\omega}_\ell\right)\right]$$

$$= -Z_\ell^* - \nu\nabla\omega\cdot\nabla\bar{\omega}_\ell - \alpha\omega\bar{\omega}_\ell$$

with

$$Z_\ell^* = \frac{1}{4\ell}\int d^2r\,(\nabla_r G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\,|\delta\omega(\mathbf{r})|^2$$

by an integration-by-parts in $\mathbf{r}$. The latter term represents a flux of enstrophy to scales less than ℓ.

If we try to mimic the argument of Duchon-Robert, we might assume that $\omega\in L_3$, which implies also by the Calderón-Zygmund inequality that $\mathbf{u}\in L_3$ (as in class). In that case it is easy to see that

$$\omega\bar{\omega}_\ell \xrightarrow[\ell\to 0]{L^1} \omega^2$$

$$\omega\bar{\omega}_\ell\mathbf{u} \xrightarrow[\ell\to 0]{L^1} \omega^2\mathbf{u}$$

$$\tfrac{1}{4}\overline{(\omega^2\mathbf{u})}_\ell - \tfrac{1}{4}\overline{(\omega^2)}_\ell\mathbf{u} \xrightarrow[\ell\to 0]{L^1} 0$$

since $\|\bar{f}_\ell - f\|_p \to 0$ as $\ell\to 0$ for $p\geq 1$. Thus, all terms on the left side of $(*)$ for $\nu=0$ converge in the sense of distributions as $\ell\to 0$, as follows:

$$\partial_t\left(\tfrac{1}{2}\omega\overline{\omega}_\ell\right) + \nabla_x\cdot\left[\left(\tfrac{1}{2}\omega\overline{\omega}_\ell\right)u + \tfrac{1}{4}\overline{(\omega^2 u)}_\ell - \tfrac{1}{4}\overline{(\omega^2)}_\ell u\right]$$

$$\xrightarrow[\ell\to 0]{\mathcal{D}'} \partial_t\left(\tfrac{1}{2}\omega^2\right) + \nabla_x\cdot\left(\tfrac{1}{2}\omega^2 u\right)$$

Hence for $\nu=\alpha=0$, the righthand side of $(*)$, which is simply $-Z^*_\ell$, also converges

$$-Z^*_\ell \xrightarrow[\ell\to 0]{\mathcal{D}'} -Z \equiv \partial_t\left(\tfrac{1}{2}\omega^2\right) + \nabla_x\cdot\left(\tfrac{1}{2}\omega^2 u\right).$$

<u>Remark</u>: We shall see later that the above argument in fact requires only $\omega\in L_p$, for any $p>2$.

Finally, for a spherically-symmetric filter kernel G,

$$(\nabla_r G)_\ell(r)\cdot\delta u = (G')_\ell(r)\,\hat{r}\cdot\delta u = (G')_\ell(r)\,\delta u_L(r)$$

so that, recalling that r can be at most 2π,

$$Z^*_\ell = \frac{1}{4\ell^3}\int_0^{2\pi} 2\pi r\,dr\, G'\!\left(\frac{r}{\ell}\right)\left\langle \delta u_L(r)(\delta\omega(r))^2\right\rangle_{ang}$$

$$= \frac{2\pi}{4}\int_0^{2\pi/\ell} u^2\,du\, G'(u)\left[\frac{\left\langle \delta u_L(r)(\delta\omega(r))^2\right\rangle_{ang}}{r}\right]_{r=\ell u}$$

by the change of variables $r=\ell u$.

If

$$\mathcal{D}\text{-}\lim_{r\to 0} \frac{\langle \delta u_L(\mathbf{r})(\delta w(\mathbf{r}))^2\rangle_{ang}}{2r} = S$$

exists, then a simple dominated convergence argument shows that for any nice test function $\varphi(\mathbf{x},t)$

$$\int_0^{2\pi/\ell} u^2\,du\, G'(u) \int d^2x\, dt\, \varphi(\mathbf{x},t) \left[\frac{\langle \delta u_L(\mathbf{r})(\delta w(\mathbf{r}))^2\rangle_{ang}}{2r} \right]_{r=\ell u} (\mathbf{x},t)$$

$$\xrightarrow[\ell\to 0]{} \int_0^\infty u^2\,du\, G'(u) \int d^2x\, dt\, \varphi(\mathbf{x},t)\, S(\mathbf{x},t)$$

assuming only $\int_0^\infty u^2 du\,|G'(u)| < +\infty$. Note by integration by parts that

$$2\pi \int_0^\infty u^2\,du\, G'(u) = -2 \int_0^\infty 2\pi u\,du\, G(u) = -2.$$

Hence, in the sense of distributions $Z = -S$, or

$$\mathcal{D}\text{-}\lim_{r\to 0} \frac{\langle \delta u_L(\mathbf{r})(\delta w(\mathbf{r}))^2\rangle_{ang}}{2r} = -Z.$$

Problem 3. (a) The definition $\omega = \nabla^{\perp}\cdot\mathbf{u}$ and thus $\nabla\omega = \nabla\nabla^{\perp}\cdot\mathbf{u}$ lead to

$$\nabla\overline{\omega}_\ell(\mathbf{x}) = \int d^2r \, \frac{1}{\ell^2}(\nabla_r\nabla_r^{\perp}G)_\ell(\mathbf{r})\cdot\mathbf{u}(\mathbf{x}+\mathbf{r})$$

by an integration by parts on the variable $\mathbf{r}$. Since the rapid decay of G implies that $\int d^2r\,(\nabla_r\nabla_r^{\perp}G)(\mathbf{r}) = 0$, we can subtract the term with $\mathbf{u}(\mathbf{x}+\mathbf{r})$ replaced by $\mathbf{u}(\mathbf{x})$, to obtain

$$\nabla\overline{\omega}_\ell(\mathbf{x}) = \frac{1}{\ell^2}\int d^2r\,(\nabla_r\nabla_r^{\perp}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r};\mathbf{x}).$$

(b) Starting from the standard relation for an incompressible $\mathbf{u}$

$$\omega\mathbf{u}^{\perp} = \omega\hat{\mathbf{z}}\times\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u} = \nabla\cdot(\mathbf{u}\mathbf{u}) - \nabla\left(\tfrac{1}{2}|\mathbf{u}|^2\right)$$

one obtains by coarse-graining

$$\overline{(\omega\mathbf{u}^{\perp})}_\ell = \nabla\cdot\overline{(\mathbf{u}\mathbf{u})}_\ell - \nabla\left(\overline{\tfrac{1}{2}|\mathbf{u}|^2}\right).$$

Subtracting the same relation for the coarse-grained field

$$\overline{\omega}_\ell\overline{\mathbf{u}}_\ell^{\perp} = \nabla\cdot(\overline{\mathbf{u}}_\ell\overline{\mathbf{u}}_\ell) - \nabla\left(\tfrac{1}{2}|\overline{\mathbf{u}}_\ell|^2\right)$$

gives

$$\begin{aligned}\sigma_\ell^{\perp} &= \overline{(\omega\mathbf{u}^{\perp})}_\ell - \overline{\omega}_\ell\overline{\mathbf{u}}_\ell^{\perp} \\ &= \nabla\cdot\tau_\ell(\mathbf{u},\mathbf{u}) - \nabla\left(\tfrac{1}{2}\tau_\ell(u_k,u_k)\right).\end{aligned}$$

(c) From the "shift trick" we know in general that

$$\partial_k T_\ell(u_i,u_j) = -\frac{1}{\ell}\Big[\int d^2r\, (\partial_k G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r})\, \delta u_j(\mathbf{r})$$

$$- \int d^2r\, (\partial_k G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r}) \cdot \int d^2r'\, G_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}')$$

$$- \int d^2r\, G_\ell(\mathbf{r})\, \delta u_i(\mathbf{r}) \cdot \int d^2r'\, (\partial_k G)_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}')\Big].$$

If we take $k=j$ and sum over j, we obtain the first formula for $(\nabla\cdot T_\ell(\mathbf{u},\mathbf{u}))_i$, noting that

$$\frac{1}{\ell}\int d^2r'\, (\partial_j G)_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}';\mathbf{x})$$

$$= \frac{1}{\ell}\int d^2r\, (\partial_j G)_\ell(\mathbf{r}')\, u_j(\mathbf{x}+\mathbf{r}')$$

$$= -\int d^2r\, G_\ell(\mathbf{r}')\, \nabla\cdot\mathbf{u}(\mathbf{x}+\mathbf{r}') = 0$$

by incompressibility. Finally, by taking

$$k \longrightarrow i$$
$$i \longrightarrow k$$
$$j \longrightarrow k$$

summing over k and multiplying by $\frac{1}{2}$, one obtains the second formula for $\partial_i\left(\frac{1}{2} T_\ell(u_k,u_k)\right)$.