Homework No.4, 550.696, Due April 7, 2014.

1. The global energy flux with spatial coarse-graining is defined by averaging the local flux over the flow domain, as:

$$\Pi_\ell(t) = \frac{1}{|D|}\int_D d^2x\ \Pi_\ell(\mathbf{x},t).$$

(a) Consider the special case $D = \mathbb{T}^2$ and with the filter kernel defined by periodization of a spherically-symmetric kernel G over $\mathbb{R}^2$. Prove the relation

$$\Pi_\ell(t) = \int_0^\infty dk\ P_\ell(k)\Pi(k,t)$$

with $P_\ell(k) = -\frac{d}{dk}|\hat{G}(\ell k)|^2$. *Hint:* Use the relation

$$E_{>\ell}(t) = \frac{1}{|D|}\int_D d^2x\ \frac{1}{2}|\bar{\mathbf{u}}_\ell(\mathbf{x},t)|^2 = \int_0^\infty dk\ |\hat{G}(\ell k)|^2 E(k,t)$$

which follows from the discussion in Homework #1, Problem 2(a).

(b) Show that

$$\int_0^\infty dk\ P_\ell(k) = 1, \quad P_\ell(k) = \ell P(\ell k),$$

for $P = P_1$, and $P(k) \geq 0$ for $|\hat{G}(k)|$ which is non-increasing. Plot P for the special case of a Gaussian filter kernel in 2D, $G(r) = \frac{1}{2\pi}\exp(-r^2/2)$.

2. (a) Derive the point-split enstrophy balance for the linearly-damped 2D Navier-Stokes equation

$$\partial_t(\frac{1}{2}\omega\omega') + \boldsymbol{\nabla}_x\boldsymbol{\cdot}[(\frac{1}{2}\omega\omega')\mathbf{u} + \frac{1}{4}(\omega')^2\delta\mathbf{u} - \nu\boldsymbol{\nabla}_x(\frac{1}{2}\omega\omega')]$$
$$= \frac{1}{4}\boldsymbol{\nabla}_r\boldsymbol{\cdot}[\delta\mathbf{u}|\delta\omega|^2] - \nu\boldsymbol{\nabla}_x\omega\boldsymbol{\cdot}\boldsymbol{\nabla}_x\omega' - \alpha\omega\omega',$$

with $a = a(\mathbf{x},t)$, $a' = a(\mathbf{x}+\mathbf{r},t)$, and $\delta a = a' - a$ for any field $a(\mathbf{x},t)$. State appropriate assumptions under which this equation is valid for $\nu = 0$ in the sense of distributions as a function of $(\mathbf{x},\mathbf{r},t)$.

(b) If the above equation is smeared with a smooth test function $G_\ell(\mathbf{r}) = \ell^{-2}G(\mathbf{r}/\ell)$ in the variable $\mathbf{r}$, then derive the formula

$$Z_\ell^*(\mathbf{x},t) = \frac{1}{4\ell}\int d^2r\ (\boldsymbol{\nabla}_r G)_\ell(\mathbf{r})\boldsymbol{\cdot}\delta\mathbf{u}(\mathbf{r};\mathbf{x},t)|\delta\omega(\mathbf{r};\mathbf{x},t)|^2$$

for the enstrophy flux. State conditions for the case $\nu = \alpha = 0$ (2D Euler) under which the limit exists

$$Z(\mathbf{x},t) = \mathcal{D} - \lim_{\ell\to 0} Z^*_\ell(\mathbf{x},t),$$

analogous to those of Duchon-Robert (2000), for 3D energy flux. Show then that

$$\mathcal{D} - \lim_{r\to 0} \frac{1}{r}\langle \delta u_L(\mathbf{r};\mathbf{x},t)|\delta\omega(\mathbf{r};\mathbf{x},t)|^2\rangle_{\rm ang} = -4Z(\mathbf{x},t)$$

for the angle-average $\langle\cdot\rangle_{\rm ang}$ over $\hat{\mathbf{r}}$, whenever the limit on the left exists.

Remark: Unfortunately, we shall see that this formulation of the 2D enstrophy dissipative anomaly probably only makes sense when $Z \equiv 0$!

3. (a) Show that

$$\boldsymbol{\nabla}\bar{\omega}_\ell = \frac{1}{\ell^2}\int d^2r\ (\boldsymbol{\nabla}_r\boldsymbol{\nabla}_r^\perp G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r}).$$

(b) For $\boldsymbol{\sigma}_\ell \equiv \tau_\ell(\omega,\mathbf{u})$, show that

$$\boldsymbol{\sigma}_\ell^\perp = \boldsymbol{\nabla}\boldsymbol{\cdot}\tau_\ell(\mathbf{u},\mathbf{u}) - \boldsymbol{\nabla}\left(\frac{1}{2}\tau_\ell(u_k,u_k)\right).$$

(c) Show using the "shift trick" from Turbulence I that

$$(\boldsymbol{\nabla}\boldsymbol{\cdot}\tau_\ell(\mathbf{u},\mathbf{u}))_i = -\frac{1}{\ell}\left[\int d^2r\ (\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\delta u_j(\mathbf{r}) - \int d^2r\ (\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\cdot\int d^2r'\ G_\ell(\mathbf{r}')\delta u_j(\mathbf{r}')\right]$$

and that

$$\partial_i\left(\frac{1}{2}\tau_\ell(u_k,u_k)\right) = -\frac{1}{2\ell}\left[\int d^2r\ (\partial_i G)_\ell(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2 - 2\int d^2r\ (\partial_i G)_\ell(\mathbf{r})\delta u_k(\mathbf{r})\cdot\int d^2r'\ G_\ell(\mathbf{r}')\delta u_k(\mathbf{r}')\right].$$