Homework No.3, 550.696, Due March 24, 2014.

1. Assuming statistical homogeneity and isotropy, the Wiener-Khinchin theorem relating the 2-point correlation of vorticity $B_2^\omega(r) = \langle \omega(\mathbf{r})\omega(\mathbf{0})\rangle$ and the enstrophy spectrum $\Omega(k)$ in 2D becomes $B_2^\omega(r) = 2\int_0^\infty J_0(kr)\Omega(k)\,dk$. For the Kraichnan-Batchelor (KB) enstrophy cascade range this yields

$$B_2^\omega(r) = 2C\eta^{2/3}\int_{k_f}^{k_{uv}} J_0(kr)\,\frac{dk}{k},$$

with the simplification that $\Omega(k) = 0$ for $k > k_{uv}$.

(a) Show that for $k_f \ll r^{-1} \ll k_{uv}$,

$$B_2^\omega(r) \sim 2C\eta^{2/3}\left[\ln 2 - \gamma - \ln(k_f r) + O\left((k_f r)^2\right)\right],$$

where γ is the Euler-Mascheroni constant, so that $\ln 2 - \gamma \doteq 0.1159$.

Hint: Use Abramowitz & Stegun, identity 11.1.20.

(b) Show that for $r^{-1} \gg k_{uv} \gg k_f$

$$B_2^\omega(r) \sim 2C\eta^{2/3}\left[\ln(k_{uv}/k_f) - \sum_{n=1}^\infty b_n (k_{uv}r)^{2n}\right]$$

with expansion coefficients $b_n = \frac{(-1)^{n+1}}{(n!)^2 2^{2n+1} n}$.

(c) Using the definition of the structure function $S_2^\omega(r) = 2[B_2^\omega(0) - B_2^\omega(r)]$ and (a),(b), show that for $k_f \ll r^{-1} \ll k_{uv}$,

$$S_2^\omega(r) \sim 4C\eta^{2/3}\left[\ln(k_{uv}r) - \ln 2 + \gamma + O\left((k_f r)^2\right)\right],$$

while for $r^{-1} \gg k_{uv} \gg k_f$

$$S_2^\omega(r) \sim 4C\eta^{2/3}\sum_{n=1}^\infty b_n (k_{uv}r)^{2n} \sim \frac{1}{2}C\eta^{2/3}(k_{uv}r)^2.$$

2. Assuming statistical homogeneity and isotropy, the Wiener-Khinchin theorem relating the 2-point correlation of velocity $B_2^{\mathbf{u}}(r) = \langle \mathbf{u}(\mathbf{r})\cdot\mathbf{u}(\mathbf{0})\rangle$ and the energy spectrum $E(k)$ in 2D becomes likewise $B_2^{\mathbf{u}}(r) = 2\int_0^\infty J_0(kr)E(k)\,dk$. For the KB enstrophy cascade range this yields

$$B_2^{\mathbf{u}}(r) = 2C\eta^{2/3}\int_{k_f}^{k_{uv}} J_0(kr)\,\frac{dk}{k^3}.$$

(a) Show that for $k_f \ll r^{-1} \ll k_{uv}$,

$$\begin{aligned} B_2^{\mathbf{u}}(r) &= C\eta^{2/3} r^2 \left[\frac{J_0(k_f r)}{(k_f r)^2} - \frac{J_1(k_f r)}{2 k_f r} - \frac{1}{2}(\ln 2 - \gamma) + \frac{1}{2}\ln(k_f r) \right] \\ &\sim C\eta^{2/3} \left[k_f^{-2} - \beta r^2 + \frac{1}{2} r^2 \ln(k_f r) + O(k_f^2 r^4) \right], \end{aligned}$$

with $\beta = \frac{1}{2}(1 + \ln 2 - \gamma) \doteq 0.5580$, and for $r^{-1} \gg k_{uv} \gg k_f$

$$B_2^{\mathbf{u}}(r) \sim C\eta^{2/3} \left[k_f^{-2} - \frac{1}{2} r^2 \ln(k_{uv}/k_f) + O\left(k_{uv}^2 r^4\right) \right]$$

Hint: $\int \frac{J_0(x)}{x^3} dx = -\frac{J_0(x)}{2x^2} + \frac{J_1(x)}{4x} - \frac{1}{4} \int \frac{J_0(x)}{x} dx$.

(b) Show using the definition of the structure function $S_2^{\mathbf{u}}(r) = 2[B_2^{\mathbf{u}}(0) - B_2^{\mathbf{u}}(r)]$ and part (a) that for $k_f \ll r^{-1} \ll k_{uv}$,

$$S_2^{\mathbf{u}}(r) \sim 2C\eta^{2/3} \left[\beta r^2 - \frac{1}{2} r^2 \ln(k_f r) + O(k_f^2 r^4) \right],$$

and for $r^{-1} \gg k_{uv}$

$$S_2^{\mathbf{u}}(r) \sim C\eta^{2/3} r^2 \ln(k_{uv}/k_f) + O\left(k_{uv}^2 r^4\right).$$

(c) Show using the definition of the 2nd-order, 2nd-difference structure function

$$S_2^{2,\mathbf{u}}(r) = 6B_2^{\mathbf{u}}(0) + 2B_2^{\mathbf{u}}(2r) - 8B_2^{\mathbf{u}}(r)$$

and part (a) that for $k_f \ll r^{-1} \ll k_{uv}$,

$$S_2^{2,\mathbf{u}}(r) \sim 4C\eta^{2/3} (\ln 2) r^2 + O(k_f^2 r^4),$$

and for $r^{-1} \gg k_{uv} \gg k_f$

$$S_2^{\mathbf{u}}(r) = O\left(k_{uv}^2 r^4\right).$$