Homework #2

Problem 1 (a) It is enough to check the result only for characteristic functions of intervals, $\varphi(x) = 1_{[c,d]}(x)$, because their linear superpositions (the simple functions) are dense in $L^1[0,1]$. Thus, one can recover the result for general $\varphi \in L^1[0,1]$ by a triangle argument. Next note that

$$\int_0^1 dx\, 1_{[c,d]}(x)\, w_n(x) = \int_c^d dx\, w_n(x)$$

$$= (d-c)\left(\frac{a+b}{2}\right) - O\left(\frac{1}{n}\max\{|a|,|b|\}\right)$$

because the subsets of $[c,d]$ with levels a and b have equal lengths, except for end interval of lengths $O(\frac{1}{n})$. Thus,

$$\lim_{n\to\infty} \int_0^1 dx\, 1_{[c,d]}(x)\, w_n(x)$$

$$= (d-c)\left(\frac{a+b}{2}\right)$$

$$= \int_0^1 dx\, 1_{[c,d]} \cdot \left(\frac{a+b}{2}\right)$$

Hence,

$$w^*\text{-}\lim_{n\to\infty} w_n(x) = \bar{w}(x) = \frac{a+b}{2}, \quad \forall x \in [0,1]$$

(b) Since

$$f(w_n(x)) = \begin{cases} f(a) & [\![nx]\!] \text{ even} \\ f(b) & [\![nx]\!] \text{ odd} \end{cases}$$

one can apply exactly the same argument as in (a) to show that

$$\lim_{n\to\infty} \int_0^1 dx\, \varphi(x)\, f(w_n(x))$$

$$= \int_0^1 dx\, \varphi(x) \cdot \left(\frac{f(a)+f(b)}{2} \right)$$

$$= \int_0^1 dx\, \varphi(x) \int \nu_x(dw)\, f(w)$$

with

$$\nu_x = \frac{1}{2}[\delta_a + \delta_b] \quad \forall x \in [0,1],$$

an equal mixture of two delta function measures at levels a and b. Thus, this is the limiting Young measure, which, in this case, is independent of the point $x \in [0,1]$.

Problem 2. (a) If $D_0 \subset D$ is the subset where $w_0(x) = b$, then

$$g(w) = \int_D d^2x \, \delta(w_0(x) - w)$$

$$= \int_{D_0} d^2x \, \delta(b - w) + \int_{D \setminus D_0} d^2x \, \delta(w)$$

$$= A_0 \, \delta(w - b) + (A - A_0)\delta(w).$$

The limiting Young measure can have only two atoms, at values 0 and b. Thus,

$$\rho(x, w) = e(x)\delta(w - b) + f(x)\delta(w)$$

but the condition $\int \rho(x, w)\, dw = 1 \Rightarrow f(x) = 1 - e(x)$. Thus,

$$\rho(x, w) = e(x)\delta(w - b) + (1 - e(x))\delta(w).$$

Then

$$\int_D \rho(x, w)\, d^2x = \left(\int_D e(x)\, d^2x \right) \delta(w - b)$$

$$+ \left(A - \int_D e(x)\, d^2x \right) \delta(w) = g(w)$$

if and only if

$$\int_D e(x)\, d^2x = A_0.$$

(b) With the approximate delta function

$$\delta_\epsilon(\omega) = \frac{1_{[-\epsilon/2,+\epsilon/2]}(\omega)}{\epsilon},$$

one can easily calculate

$$\ln \rho_\epsilon(x,\omega) = \ln e(x)\, \delta_\epsilon(\omega-b) + \ln(1-e(x))\, \delta_\epsilon(\omega) - \ln\epsilon$$

and

$$\rho_\epsilon(x,\omega) \ln \rho_\epsilon(x,\omega)$$

$$= \frac{1}{\epsilon}\Big[e(x) \ln e(x)\, \delta_\epsilon(\omega-b) + (1-e(x)) \ln(1-e(x))\, \delta_\epsilon(\omega) \Big]$$

$$- \rho_\epsilon(x,\omega) \ln\epsilon .$$

Thus,

$$S[\rho_\epsilon] = -\int_D d^2x \int d\omega\, \rho_\epsilon(x,\omega) \ln \rho_\epsilon(x,\omega)$$

$$= \frac{1}{\epsilon} S[e] + \ln\epsilon$$

with

$$S[e] = -\int_D d^2x \left[e(x) \ln e(x) + (1-e(x)) \ln(1-e(x)) \right).$$

(c) Carrying out the variation with the Lagrange multipliers

$$\delta S - \alpha\, \delta A_0 - \frac{\beta}{b}\, \delta E = 0$$

one obtains

$$\int_D d^2x \left[-\ln\left(\frac{e(\mathbf{x})}{1-e(\mathbf{x})} \right) - \alpha + \beta\psi(\mathbf{x}) \right] \delta e(\mathbf{x}) = 0$$

with

$$\psi(\mathbf{x}) \equiv b \int_D d^2y\, G(\mathbf{x},\mathbf{y})\, e(\mathbf{y})$$

so that $\Delta\psi(\mathbf{x}) = b\,e(\mathbf{x}) = \bar{\omega}(\mathbf{x})$. The variational equation is then easily solved to give

$$e(\mathbf{x}) = \frac{\exp\left(-\alpha + \beta\psi(\mathbf{x})\right)}{1 + \exp\left(-\alpha + \beta\psi(\mathbf{x})\right)}.$$

(d) Note that

$$A_0 = \int_D d^2x\, e(\mathbf{x}) = e^{-\alpha} \int \frac{e^{\beta\psi(\mathbf{x})}}{1 + e^{-\alpha+\beta\psi(\mathbf{x})}}\, d^2x$$

$$\underset{\alpha\to\infty}{\sim} e^{-\alpha} \int e^{\beta\psi(\mathbf{x})}\, d^2x$$

if one assumes that

$$\int e^{\beta \psi(x)}\, d^2x < +\infty$$

as $\alpha \to \infty$. Hence, the condition

$$A_0 b = 1 \implies b e^{-\alpha} \sim \frac{1}{\left(\int e^{\beta\psi(x)}\, d^2x\right)}, \quad \alpha \to \infty$$

In that case

$$b e(x) = b e^{-\alpha} \cdot \frac{e^{\beta\psi(x)}}{1 + e^{-\alpha + \beta\psi(x)}}$$

$$\xrightarrow[\alpha \to \infty]{} \frac{1}{\int e^{\beta\psi(x)}\, d^2x} \cdot \frac{e^{\beta\psi(x)}}{1+0} = \frac{e^{\beta\psi(x)}}{Z}$$

with $Z = \int e^{\beta\psi(x)}\, d^2x$ and, in the limit, the stream function satisfies

$$\Delta\psi(x) = \frac{e^{\beta\psi(x)}}{Z}.$$

This is exactly the prediction of the Onsager-Joyce-Montgomery point-vortex model.

Problem 3. (a) Recall the local balance equation for the invariant I_h:

$$\partial_t h(\omega) + \nabla \cdot \left[h(\omega)\mathbf{u} - \nu \nabla h(\omega) \right] = -\nu h''(\omega)|\nabla\omega|^2 + h'(\omega) q .$$

Because of the homogeneity and stationarity of the forcing and assuming a unique invariant measure, the stationary ensemble must inherit these symmetries. Hence, the averages of the $\partial_t(\cdot)$ and $\nabla\cdot(\cdot)$ terms vanish, and one obtains

$$\eta_h = \langle h'(\omega) q \rangle = \langle \nu h''(\omega) |\nabla\omega|^2 \rangle = \eta_h^{(\nu)} .$$

(b) Gaussian integration-by-parts identity gives

$$\langle h'(\omega) q \rangle = 2 \int_{-\infty}^{+\infty} dt' \int d^2x' \, Q(\mathbf{x}-\mathbf{x}', t-t') \times \left\langle \frac{\delta}{\delta q(\mathbf{x}',t')} h'(\omega(\mathbf{x},t)) \right\rangle$$

$$= 2 \int_{-\infty}^{t} dt' \int d^2x \, Q(\mathbf{x}-\mathbf{x}', t-t') \times \left\langle h''(\omega(\mathbf{x},t)) \frac{\delta \omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} \right\rangle .$$

We have invoked causality of the response function to restrict the integration range to $t' < t$.

(c) From the equation of motion

$$\partial_t \left(\frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} \right) + \mathbf{u}\cdot\nabla \left(\frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} \right) + \frac{\delta\mathbf{u}(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} \cdot \nabla\omega$$

$$= \nu \Delta_x \left(\frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} \right) + \delta^2(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$

Integrating over time t, one thus obtains for $t > t'$

$$\frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} = \delta^2(\mathbf{x}-\mathbf{x}') + O((t-t'))$$

since by causality $\frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')} = 0$ for $t < t'$. If we assume a white-noise force

$$Q(\mathbf{x}-\mathbf{x}', t-t') = Q(\mathbf{x}-\mathbf{x}')\,\delta(t-t'),$$

then the integration-by-parts formula becomes

$$\langle h'(\omega) q \rangle = 2 \int_{-\infty}^{t} dt' \int d^2x\; Q(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \times \left\langle h''(\omega(\mathbf{x},t)) \underbrace{\frac{\delta\omega(\mathbf{x},t'+)}{\delta q(\mathbf{x}',t')}}_{\delta^2(\mathbf{x}-\mathbf{x}')} \right\rangle$$

$$= 2\cdot\frac{1}{2} Q(0) \langle h''(\omega(\mathbf{x},t)) \rangle$$

$$= Q(0) \langle h''(\omega) \rangle \quad \text{by the "}\tfrac{1}{2}\text{-delta" rule}$$

On the other hand, for $h(\omega) = \frac{1}{2}\omega^2$ this reduces to

$$\eta = \langle \omega q \rangle = Q(0) \cdot \langle 1 \rangle = Q(0).$$

Hence, we can rewrite the result as

$$\eta_h = \langle h''(\omega) \rangle \eta .$$

(d) Combining (a) & (c) gives

$$\langle h''(\omega)\, \nu |\nabla \omega|^2 \rangle = \eta_h^{(\omega)} = \eta_h$$

$$= \langle h''(\omega) \rangle \eta = \langle h''(\omega) \rangle \langle \nu |\nabla \omega|^2 \rangle.$$

Since we can take $h''(\omega) = f(\omega)$ to be any measurable function, one obtains by the definition of conditional expectation as a Radon–Nikodym derivative that

$$\langle f(\omega)\, \nu |\nabla \omega|^2 \rangle = \langle f(\omega) \cdot \langle \nu |\nabla \omega|^2 \rangle \rangle$$

implies

$$\langle \nu |\nabla \omega|^2 \,|\, \omega \rangle = \langle \nu |\nabla \omega|^2 \rangle.$$

More formally, one can take $f(\omega)$ to be a smooth function converging to a delta function $\delta(\omega - \omega_0)$, so that

$$\langle \nu |\nabla\omega|^2 | \omega=\omega_0 \rangle \equiv \frac{\langle \nu |\nabla\omega|^2 \delta(\omega-\omega_0) \rangle}{\langle \delta(\omega-\omega_0) \rangle}$$

$$= \langle \nu |\nabla\omega|^2 \rangle .$$

(e) The result obtained in (d) is quite remarkable, but it clearly depends upon the precise details of the forcing. There is no reason in general for the result to hold for arbitrary forcing that

$$\eta_h = \langle h''(\omega) \rangle \eta ,$$

which is the basis for the result in (d). Thus, it is very unlikely that the statistical relation $\langle \nu |\nabla\omega|^2 | \omega \rangle = \langle \nu |\nabla\omega|^2 \rangle$ holds in general. For related remarks, see Kraichnan (1967), p. 1422, 2nd column. Of course, this does not rule out that some particular statistic (say, the large-amplitude tail of the PDF $P(\xi)$ of $\xi = |\nabla\omega|$) might be universal for some reason.

Problem 4. (a) Since $|\mathbf{k}| > k_0$ for $|\mathbf{k}| \neq 0$, one gets for $p > 2$ that

$$\eta_{ir} = \alpha_p \sum_{\mathbf{k} \neq 0} \frac{\langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle}{|\mathbf{k}|^{p-2}}$$

$$\leq \frac{\alpha_p}{k_0^{p-2}} \sum_{\mathbf{k} \neq 0} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \frac{2\alpha_p E}{k_0^{p-2}} = k_0^2 \left(\frac{2\alpha_p E}{k_0^p} \right).$$

Then using $\eta \cong C \varepsilon k_f^2$

$$\frac{\eta_{ir}}{\eta} \leq (\text{const.}) \left(\frac{k_0}{k_f} \right)^2 \left(\frac{\alpha_p E}{\varepsilon k_0^p} \right) \ll 1$$

when k_f is chosen much greater than $k_0 \left(\frac{\alpha_p E}{\varepsilon k_0^p} \right)^{1/2}$.

(b) We use the same argument here as for the case without large-scale damping:

$$\varepsilon_{uv} = \sum_{\mathbf{k}} \nu |\mathbf{k}|^2 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

is bounded by Cauchy-Schwartz inequality as

$$\varepsilon_{uv} \leq \nu \sqrt{\sum_{\mathbf{k}} |\mathbf{k}|^4 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \sum_{\mathbf{k}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle}$$

$$= \nu \sqrt{\frac{\eta}{\nu} \cdot 2E}$$

$$= (2\nu\eta E)^{1/2}$$

and

$$\frac{\varepsilon_{uv}}{\varepsilon} \leq \left(\frac{2\nu\eta E}{\varepsilon^2} \right)^{1/2} = (\text{const.}) \left(\frac{\nu k_f^2 E}{\varepsilon} \right)^{1/2}$$

using again $\eta = C k_f^2 \varepsilon$.

(c) The derivation of the balance equation is straightforward, exactly as in the classnotes for energy flux, but multiplying first by $|\mathbf{k}|^2$ before summing over $\mathbf{k}$. Note then that

$$\sum_{|\mathbf{k}|<K} \alpha_p |\mathbf{k}|^{2-p} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \sum_{\mathbf{k}\neq 0} \alpha_p |\mathbf{k}|^{2-p} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$= \eta_{ir} = \eta \cdot O\left(\left(\frac{k_0}{k_f}\right)^2 \frac{\alpha_p E}{\varepsilon k_0^p} \right)$$

by part (a).

Also,

$$\sum_{|\mathbf{k}|<K} \nu |\mathbf{k}|^4 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \nu K^4 \sum_{|\mathbf{k}|<K} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \nu K^4 \sum_{\mathbf{k}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$= \nu K^4 E = \eta \cdot O\left(\frac{\nu K^4 E}{\varepsilon k_f^2} \right)$$

(d) It was shown in the course notes that

$$-\Pi(K) = \sum_{|\mathbf{k}|<K} D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle , \quad K < k_f$$

with $D(\mathbf{k}) = \alpha_p |\mathbf{k}|^{-p} + \nu |\mathbf{k}|^2$. Using the global balance

$$\varepsilon = \sum_{\mathbf{k}} D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

this can be written as

$$-\Pi(K) = \varepsilon - \sum_{|\mathbf{k}| \geq K} D(\mathbf{k}) \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle .$$

Next, note that

$$\sum_{|\mathbf{k}| \geq K} \alpha_p |\mathbf{k}|^{-p} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \frac{1}{K^2} \sum_{|\mathbf{k}| \geq K} \alpha_p |\mathbf{k}|^{2-p} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \frac{1}{K^2} \sum_{\mathbf{k} \neq 0} \alpha_p |\mathbf{k}|^{2-p} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$= \frac{\eta_{ir}}{K^2}$$

$$= O\left(\left(\frac{k_0}{K}\right)^2 \frac{\alpha_p E}{\varepsilon k_0^p} \right)$$

by the argument of (a). Of course,

$$\sum_{|\mathbf{k}| \geq K} \nu |\mathbf{k}|^2 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$\leq \sum_{\mathbf{k}} \nu |\mathbf{k}|^2 \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle$$

$$= \varepsilon_{uv} = \varepsilon \cdot O\left(\left(\nu k_f^2 \frac{E}{\varepsilon} \right)^{1/2} \right)$$

by part (b).

(e) For linear damping ($p=0$),

$$\eta_{ir} = \sum_{\mathbf{k}} \alpha |\mathbf{k}|^2 \langle |\hat{u}(\mathbf{k})|^2 \rangle$$

$$= \alpha \Omega$$

Hence, using $\eta = C k_f^2 \varepsilon$ again

$$\frac{\eta_{ir}}{\eta} = \text{const.} \frac{\alpha \Omega}{k_f^2 \varepsilon}$$

$$= \text{const.} \left(\frac{k_0}{k_f} \right)^2 \frac{\alpha \Omega}{k_0^2 \varepsilon}$$

$$\ll 1$$

if one chooses k_f much greater than $k_0 \left(\frac{\alpha \Omega}{\varepsilon k_0^2} \right)^{1/2}$.