Homework No.2, 550.696, Due March 10, 2014.

1. Consider the sequence of fields $\omega_n(x)$ with $x \in [0,1]$ defined for $n = 1, 2, 3, ...$ by $\omega_n(x) = a + (b-a)\text{mod}(\llbracket nx \rrbracket, 2)$, for two real numbers $a < b$. Here $\llbracket \cdot \rrbracket$ denotes the integer part of a real number, so that

$$\omega_n(x) = \begin{cases} a & \text{for } \llbracket nx \rrbracket \text{ even} \\ b & \text{for } \llbracket nx \rrbracket \text{ odd} \end{cases}$$

(a) Show that the sequence $\omega_n(x) \in L^\infty[0,1]$ converges in weak-$*$ sense as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^1 dx\ \varphi(x)\omega_n(x) = \int_0^1 dx\ \varphi(x)\bar{\omega}(x), \quad \forall \varphi \in L^1[0,1],$$

and identify the limit $\bar{\omega}(x)$. Notice that it is enough to check convergence for characteristic functions of intervals $\varphi(x) = 1_{[c,d]}(x)$, $[c,d] \subset [0,1]$.

(b) Using the results of (a), show that for any continuous function f on $[a,b]$

$$\lim_{n\to\infty} \int_0^1 dx\ \varphi(x) f(\omega_n(x)) = \int_0^1 dx\ \varphi(x) \int \nu_x(d\omega) f(\omega)$$

and identify the limiting Young measure ν_x.

2. Consider a domain $D \subset \mathbb{R}^2$ with finite area A and an initial condition for the 2D Euler equation inside D which consists of a finite number of vortex patches with constant level b whose total area is $A_0 < A$ and, outside those patches, zero vorticity.

(a) Calculate the invariant function

$$g(\omega) = \int_D d^2x\ \delta(\omega_0(\mathbf{x}) - \omega)$$

corresponding to this initial data and explain why the only limiting Young measure that can satisfy the constraint $\int_D d^2x\ \rho(\mathbf{x},\omega) = g(\omega)$ must have the form

$$\rho(\mathbf{x},\omega) = e(\mathbf{x})\delta(\omega - b) + (1 - e(\mathbf{x}))\delta(\omega)$$

for a nonnegative measurable function $e(\mathbf{x})$ satisfying

$$\int_D d^2x\ e(\mathbf{x}) = A_0. \qquad (*)$$

(b) Show by approximating

$$\rho_\epsilon(\mathbf{x},\omega) = e(\mathbf{x}) \cdot \frac{1}{\epsilon} 1_{[\omega_0-\epsilon/2,\omega_0+\epsilon/2]}(\omega) + (1-e(\mathbf{x})) \cdot \frac{1}{\epsilon} 1_{[-\epsilon/2,+\epsilon/2]}(\omega)$$

that

$$S[\rho_\epsilon] = -\frac{1}{\epsilon}\int_D d^2x\ [e(\mathbf{x})\ln e(\mathbf{x}) + (1-e(\mathbf{x}))\ln(1-e(\mathbf{x}))] + \ln(\epsilon).$$

Thus, the maximum entropy state can be obtain by maximizing

$$S[e] = -\int_D d^2x\ [e(\mathbf{x})\ln e(\mathbf{x}) + (1-e(\mathbf{x}))\ln(1-e(\mathbf{x}))].$$

Remark: This may also be shown by a direct counting argument analogous to Fermi-Dirac statistics for quantum ensembles.

(c) Maximizing $S[e]$ with Lagrange multiplier α for the area constraint (*) and with Lagrange multiplier β for the constraint of constant (rescaled) energy

$$E[e]/b = -\frac{b}{2}\int_D d^2x \int_D d^2y\ G(\mathbf{x},\mathbf{y})e(\mathbf{x})e(\mathbf{y}) = E_0/b,$$

show that the maximum-entropy solution is

$$e(\mathbf{x}) = \frac{\exp(-\alpha+\beta\psi(\mathbf{x}))}{1+\exp(-\alpha+\beta\psi(\mathbf{x}))}$$

with $\triangle\psi(\mathbf{x}) = \bar{\omega}(\mathbf{x}) = b\cdot e(\mathbf{x})$.

(d) In the "dilute vorticity" limit $A_0 \to 0$ with $A_0 b = 1$ show that the above theory recovers the predictions of the Onsager-Joyce-Montgomery point-vortex model.

3. We consider here the problem of the forced 2D NS equation on $\mathbb{T}^2$ in vorticity formulation

$$\partial_t\omega + \mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = \nu\triangle\omega + q$$

for q a homogeneous and stationary Gaussian random field with zero mean and covariance $\langle q(\mathbf{x},t)q(\mathbf{x}',t')\rangle = 2Q(\mathbf{x}-\mathbf{x}',t-t')$. We focus on the properties of the invariants

$$I_h(t) = \int_{\mathbb{T}^2} d^2x\ h(\omega(\mathbf{x},t))$$

for any $h \in C^2$.

(a) Derive the balance relation $\eta_h = \eta_h^{(\nu)}$ between the mean input rate $\eta_h = \langle h'(\omega)q\rangle$ and the mean viscous dissipation rate $\eta_h^{(\nu)} = \nu\langle h''(\omega)|\boldsymbol{\nabla}\omega|^2\rangle$ of the above invariant.

(b) Use the Gaussian-integration-by-parts identity to derive a formula for the mean input rate $\eta_h = \langle h'(\omega)q\rangle$ in terms of the covariance Q and the vorticity response function

$$R(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\omega(\mathbf{x},t)}{\delta q(\mathbf{x}',t')}.$$

(c) When q is *white-noise in time*, show that your formula in (b) reduces to the formula $\eta_h = \langle h''(\omega)\rangle\eta$, where η is the input rate of enstrophy.

(d) Using the balance of input rates and viscous dissipation in (a) and your result in (c), show the following remarkable property holds for the statistical stationary state with white-noise forcing:

$$\langle\nu|\boldsymbol{\nabla}\omega|^2|\omega\rangle = \langle\nu|\boldsymbol{\nabla}\omega|^2\rangle,$$

i.e. the mean of the enstrophy dissipation conditioned on any vorticity level ω is equal to the unconditional mean.

(e) Do you think this relation is likely to hold for other forcing functions q that are not white-noise? What does this suggest to you about the universality of all enstrophy dissipation statistics in the 2D enstrophy cascade?

4. Consider the statistical stationary state of 2D NS in $\mathbb{T}^2 = [0,L]^2$ with an additional large-scale damping

$$\partial_t\omega + \mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\omega = \nu\triangle\omega - \alpha_p(-\triangle)^{-p/2}\omega + q$$

and forcing q at wavenumbers $\simeq k_f$ which inputs energy at rate ε and enstrophy at rate $\eta = (const.)\varepsilon k_f^2$. We make the following assumption:

Hypothesis: The mean energy stays bounded by a fixed number E if one takes first $k_f \gg k_0 = 2\pi/L$ and then $\nu \ll \varepsilon/(k_f^2 E)$, with other parameters (i.e. $\alpha_p, k_0, \varepsilon$) fixed.

Define the various mean dissipations:

$$\varepsilon_{ir} = \sum_{\mathbf{k}\neq\mathbf{0}} \alpha_p|\mathbf{k}|^{-p}\langle|\hat{\mathbf{u}}(\mathbf{k})|^2\rangle, \quad \varepsilon_{uv} = \sum_{\mathbf{k}} \nu|\mathbf{k}|^2\langle|\hat{\mathbf{u}}(\mathbf{k})|^2\rangle,$$

$$\eta_{ir} = \sum_{\mathbf{k}\neq\mathbf{0}} \alpha_p|\mathbf{k}|^{2-p}\langle|\hat{\mathbf{u}}(\mathbf{k})|^2\rangle, \quad \eta_{uv} = \sum_{\mathbf{k}} \nu|\mathbf{k}|^4\langle|\hat{\mathbf{u}}(\mathbf{k})|^2\rangle.$$

(a) Show that for $p > 2$,

$$\frac{\eta_{ir}}{\eta} \le (\text{const.})\left(\frac{k_0}{k_f}\right)^2 \frac{\alpha_p E}{\varepsilon k_0^p} \ll 1, \quad \eta_{uv} \approx \eta.$$

(b) Show that

$$\frac{\varepsilon_{uv}}{\varepsilon} \le (\text{const.})\left(\frac{\nu k_f^2 E}{\varepsilon}\right)^{1/2} \ll 1, \quad \varepsilon_{ir} \approx \varepsilon.$$

(c) Show that the spectral enstrophy balance relation holds

$$Z(K) = \eta - \sum_{|\mathbf{k}|<K} |\mathbf{k}|^2 \left[\alpha_p |\mathbf{k}|^{-p} + \nu |\mathbf{k}|^2\right] \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle, \quad K > k_f$$

Use this relation to show that for $p > 2$

$$Z(K) = \eta \left[1 + O\left(\left(\frac{k_0}{k_f}\right)^2 \frac{\alpha_p E}{\varepsilon k_0^p}\right) + O\left(\nu K^4 \frac{E}{k_f^2 \varepsilon}\right)\right]$$

and thus $Z(K) \approx \eta$ at least over the range

$$k_f \ll K \ll k_f \left(\frac{\varepsilon}{\nu k_f^2 E}\right)^{1/4}.$$

(d) Show that the spectral energy balance relation holds

$$-\Pi(K) = \varepsilon - \sum_{|\mathbf{k}|\geq K} \left[\alpha_p |\mathbf{k}|^{-p} + \nu |\mathbf{k}|^2\right] \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle, \quad K < k_f$$

Use this relation to show that

$$-\Pi(K) = \varepsilon \left[1 + O\left(\left(\frac{k_0}{K}\right)^2 \frac{\alpha_p E}{\varepsilon k_0^p}\right) + O\left(\left(\nu k_f^2 \frac{E}{\varepsilon}\right)^{1/2}\right)\right]$$

and thus $\Pi(K) \approx -\varepsilon$ at least over the range

$$k_0 \left(\frac{\alpha_p E}{\varepsilon k_0^p}\right)^{1/2} \ll K \ll k_f$$

(e) For the case of *linear Ekman damping* ($p = 0$), show that (a) is replaced by

$$\frac{\eta_{ir}}{\eta} \leq (\text{const.}) \left(\frac{k_0}{k_f}\right)^2 \frac{\alpha_p \Omega}{\varepsilon k_0^2} \ll 1, \quad \eta_{uv} \approx \eta$$

if the main hypothesis is valid for enstrophy Ω as well as energy E. (We shall see later that this is true for linear damping!)