Homework #1

Problem 1. (a) Clearly

$$\nabla\times(\mathcal{L}\mathbf{u}) = \nabla\times\left(-\hat{\mathbf{z}}\times\mathbf{u} + \cancel{\nabla q}^{\,0}\right)$$
$$= (\hat{\mathbf{z}}\cdot\nabla)\mathbf{u}$$

by vector calculus identities. Likewise,

$$-\hat{\mathbf{z}}\times(\nabla\times\mathbf{u}) = (\hat{\mathbf{z}}\cdot\nabla)\mathbf{u} - \nabla u_z .$$

Since $(\hat{\mathbf{z}}\cdot\nabla)\mathbf{u}$ is divergence-free, we see that $q = u_z$ and

$$\mathcal{L}(\nabla\times\mathbf{u}) = -\hat{\mathbf{z}}\times(\nabla\times\mathbf{u}) + \nabla u_z$$
$$= (\hat{\mathbf{z}}\cdot\nabla)\mathbf{u}$$
$$= \nabla\times(\mathcal{L}\mathbf{u}). \checkmark$$

(b) Since $\mathbf{k}\times\hat{\mathbf{z}}^{\perp} = \mathbf{k}\times\hat{\mathbf{z}}$

$$i\mathbf{k}\times h_s(\mathbf{k}) = i\mathbf{k}\times\hat{\mathbf{z}} - s\,\mathbf{k}\times(\hat{\mathbf{k}}\times\hat{\mathbf{z}})$$
$$= s|\mathbf{k}|\left[is\,\hat{\mathbf{k}}\times\hat{\mathbf{z}} - \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{z}})\right]$$
$$= s|\mathbf{k}|\left[\hat{\mathbf{z}}^{\perp} + is\,\hat{\mathbf{k}}\times\hat{\mathbf{z}}\right]$$
$$= s|\mathbf{k}|\,h_s(\mathbf{k}) \checkmark$$

Since $\hat{\mathbf{z}} \times \hat{\mathbf{z}}^{\perp} = -(\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \hat{\mathbf{z}} \times \hat{\mathbf{k}}$

$$
\begin{aligned}
-\hat{\mathbf{z}} \times \mathbf{h}_s(\mathbf{k}) &= (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \hat{\mathbf{z}} \times \hat{\mathbf{k}} - is\, \hat{\mathbf{z}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{z}}) \\
&= -(\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \hat{\mathbf{k}} \times \hat{\mathbf{z}} + is\, \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \\
&= is\, (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}}) \times \left[is\, \hat{\mathbf{k}} \times \hat{\mathbf{z}} \right] \\
&\qquad + is \left[-\hat{\mathbf{k}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \hat{\mathbf{z}} \right] \\
&= is\, (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}}) \left[\hat{\mathbf{z}} + is\, \hat{\mathbf{k}} \times \hat{\mathbf{z}} \right] - is\, \hat{\mathbf{k}} \\
&= is\, (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}}) \left[\hat{\mathbf{z}}^{\perp} + is\, \hat{\mathbf{k}} \times \hat{\mathbf{z}} \right] \\
&\qquad - is \left[1 - (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})^2 \right] \hat{\mathbf{k}} \\
&= is\, (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \mathbf{h}_s(\mathbf{k}) - ik\, \hat{\mathbf{q}}(\mathbf{k})
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathcal{L}\, \mathbf{h}_s(\mathbf{k}) &= -\hat{\mathbf{z}} \times \mathbf{h}_s(\mathbf{k}) + ik\, \hat{\mathbf{q}}(\mathbf{k}) \\
&= is\, (\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})\, \mathbf{h}_s(\mathbf{k}) . \checkmark
\end{aligned}
$$

Problem 2 (a) Simple direct calculations give

$$E_{<\ell}(t) = \frac{1}{2}\langle \tau_\ell(u_i, u_i)\rangle$$

$$= \frac{1}{2}\left[\langle |u|^2\rangle - \langle |\bar{u}_\ell|^2\rangle\right]$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left[1 - \left|(2\pi)^2\hat{G}_\ell^P(\mathbf{k})\right|^2\right]|\hat{u}(\mathbf{k},t)|^2$$

where we used

$$\bar{u}_\ell(\mathbf{x}) = \int_{\mathbb{T}^2} d^2r\, G_\ell^P(\mathbf{r})\, u(\mathbf{x}+\mathbf{r})$$

$$= (2\pi)^2 (G_\ell^P * u)(\mathbf{r}) \quad \left(\begin{array}{c}\text{assuming for}\\ \text{simplicity that}\\ G \text{ is even}\end{array}\right)$$

and then applying the convolution theorem for Fourier transforms on $\mathbb{T}^2$. Furthermore, the Poisson summation formula with our definitions of Fourier transforms in the course gives

$$(2\pi)^2\hat{G}_\ell^P(\mathbf{k}) = \hat{G}_\ell(\mathbf{k}) = \hat{G}(\ell\mathbf{k})$$

implying

$$E_{<\ell}(t) = \frac{1}{2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\left[1 - |\hat{G}(\ell\mathbf{k})|^2\right]|\hat{u}(\mathbf{k},t)|^2$$

Note that for all $\mathbf{k} \in \mathbb{R}^2$

$$|\hat{G}(\mathbf{k})| = \left| \int_{\mathbb{R}^2} d^2r \, G(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right|$$

$$\leq \int_{\mathbb{R}^2} d^2r \, G(\mathbf{r}) = 1$$

Hence,

$$\sup_{|\mathbf{k}| \geq 1} \frac{1 - |\hat{G}(\mathbf{k})|^2}{|\mathbf{k}|^2} \leq 1$$

whereas using $|\hat{G}(\mathbf{k})|^2 = 1 - O(|\mathbf{k}|^2)$, i.e. $1 - |\hat{G}(\mathbf{k})|^2 < C|\mathbf{k}|^2$,

$$\sup_{|\mathbf{k}| \leq 1} \frac{1 - |\hat{G}(\mathbf{k})|^2}{|\mathbf{k}|^2} \leq C.$$

It follows that

$$E_{<\ell}(t) = \frac{\ell^2}{2} \sum_{\mathbf{k} \in \mathbb{Z}^2} \frac{1 - |\hat{G}(\ell\mathbf{k})|^2}{|\ell\mathbf{k}|^2} \cdot |\hat{\omega}(\mathbf{k},t)|^2$$

$$\leq \ell^2 \left[\sup_{\mathbf{k} \in \mathbb{R}^2} \frac{1 - |\hat{G}(\mathbf{k})|^2}{|\mathbf{k}|^2} \right] \cdot \sum_{\mathbf{k} \in \mathbb{Z}^2} \frac{1}{2} |\hat{\omega}(\mathbf{k},t)|^2$$

$$= \ell^2 \max\{1, C\} \, \Omega(t)$$

(b) A similar calculation as in (a) shows that

$$\Omega_{>\ell}(t) = \frac{1}{2}\langle |\overline{\omega}_\ell|^2 \rangle$$

$$= \frac{1}{2}\sum_{\mathbf{k}} \left| (2\pi)^2 \hat{G}^p_\ell(\mathbf{k}) \right|^2 |\hat{\omega}(\mathbf{k},t)|^2$$

$$= \frac{1}{2}\sum_{\mathbf{k}} |\hat{G}(\ell\mathbf{k})|^2 \cdot |\mathbf{k}|^2 |\hat{\mathbf{u}}(\mathbf{k},t)|^2$$

$$= \frac{1}{2\ell^2}\sum_{\mathbf{k}} |\ell\mathbf{k}\cdot\hat{G}(\ell\mathbf{k})|^2 \, |\hat{\mathbf{u}}(\mathbf{k},t)|^2$$

Note that for all $\mathbf{k}\in\mathbb{R}^2$

$$i\mathbf{k}\,\hat{G}(\mathbf{k}) = \int_{\mathbb{R}^2} d^2x \; \nabla G(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\Longrightarrow \quad |\mathbf{k}\hat{G}(\mathbf{k})| \leq \int_{\mathbb{R}^2} d^2x \, |\nabla G(\mathbf{x})| = \|\nabla G\|_{L^1} < +\infty$$

and thus

$$\Omega_{>\ell}(t) \leq \frac{\|\nabla G\|^2_{L^1}}{\ell^2} \sum_{\mathbf{k}} \frac{1}{2}|\hat{\mathbf{u}}(\mathbf{k},t)|^2$$

$$= \frac{C}{\ell^2} E(t).$$

Problem 3 (a) The total helicity in wavenumbers $< K$ can be bounded as

$$\left| H_{<K}(t) \right| = \left| \int_0^K dk\, H(k,t) \right|$$

$$\leq \int_0^K dk\, |H(k,t)|$$

$$\leq \int_0^K dk\; 2k\, E(k,t)$$

$$\leq 2K \int_0^K dk\, E(k,t)$$

$$\leq 2K \int_0^\infty dk\, E(k,t) = 2K \cdot E(t)$$

However, this is a bit misleading, because $H(k,t)$ is a signed quantity and thus one could have subintervals $[k_1, k_2]$

$$\left| \int_{k_1}^{k_2} dk\, H(k,t) \right| \gg 2K \cdot E(t)$$

~~at~~ as long as there is another interval $[k_3, k_4] \subseteq [0, K] \setminus [k_1, k_2]$ with equal helicity of the opposite sign.

For energy there is no upper bound at all on the amount to reach wavenumbers $>K$, because there is only a lower bound

$$E_{>K}(t) = \int_K^\infty dk\, E(k,t)$$

$$\geq \int_K^\infty dk \frac{|H(k,t)|}{2k}$$

$$\geq \frac{1}{2K} \int_K^\infty dk\, |H(k,t)|$$

$$\geq \frac{1}{2K} \left| \int_K^\infty dk\, H(k,t) \right| .$$

Furthermore, $\int_K^\infty dk\, H(k,t)$ is not conserved in time, even if the lower bound were an upper bound.

(b) For 3D NS projected onto the +-helicity modes, the situation is quite different. For a mode $h_+(\mathbf{k})$

$$i\mathbf{k} \times h_+(\mathbf{k}) = k\, h_+(\mathbf{k})$$

and thus

$$i\mathbf{k} \times \mathbf{h}_+(\mathbf{k}) \cdot \mathbf{h}_+^*(\mathbf{k}) = k\,|\mathbf{h}_+(\mathbf{k})|^2 .$$

Summing over all such modes with fixed wavenumber, one obtains now that

$$H(k,t) \geq 0 \quad \text{and} \quad H(k,t) = 2k\,E(k,t) .$$

The situation is therefore now exactly analogous to 2D! One can see first of all that

$$H_{<K}(t) \leq 2K \cdot E(t)$$

and also that

$$E_{>K}(t) = \int_K^\infty dk \, \frac{H(k,t)}{2k}$$

$$\leq \frac{1}{2K} \int_0^\infty dk\, H(k,t) = \frac{1}{2K} H(t) .$$

Furthermore, $E(t) = E_0$ and $H(t) = H_0$, because both quantities are conserved in detail for individual helical modes and thus for the dynamics including all the triadic interactions with positive helicity modes.

Problem 4, (a) Since $e^{\beta\psi(r)} = \dfrac{1}{(1-Ar^2)^2}$, one finds

$$Z = \int_0^1 2\pi r\, dr\, e^{\beta\psi(r)}$$

$$= \pi \int_0^1 \frac{2r\,dr}{(1-Ar^2)^2} \qquad \text{with } u = r^2$$

$$= \pi \int_0^1 \frac{du}{(1-Au)^2} = \frac{\pi}{A} \left. \frac{1}{1-Au} \right|_{u=0}^{u=1} = \frac{\pi}{1-A}$$

Furthermore,

$$u_\theta = \frac{\partial\psi}{\partial r} = -\frac{2}{\beta} \frac{1}{1-Ar^2}(-2Ar) = \frac{4A}{\beta} \frac{r}{1-Ar^2} \checkmark$$

and thus

$$r u_\theta = r\frac{\partial\psi}{\partial r} = \frac{4}{\beta} \frac{Ar^2}{1-Ar^2} = \frac{4}{\beta}\left(-1 + \frac{1}{1-Ar^2}\right)$$

So that

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{1}{r} \cdot \frac{4}{\beta} \cdot \frac{-1}{(1-Ar^2)^2} \cdot (-2Ar)$$

$$= \frac{8A}{\beta} \frac{1}{(1-Ar^2)^2} = \frac{1-A}{\pi} \frac{1}{(1-Ar^2)^2} \checkmark$$

using $1-A = \frac{8\pi}{\beta}A$. Finally, $\omega = \Delta\psi = \frac{1}{Z}e^{\beta\psi}$.

(b) Clearly

$$w(\mathbf{x}) \geq 0$$

inside the disk D. Also,

$$\int_D w(\mathbf{x})\, d^2\mathbf{x} = \frac{\int_D e^{\beta\psi}\, d^2\mathbf{x}}{Z} = \frac{Z}{Z} = 1.$$

Furthermore, it is advantageous to write w in the form

$$w(r) = \frac{8}{|\beta|} \frac{r_1^2}{(r^2+r_1^2)^2}.$$

We then see that for any $0<\rho<1$ and $\beta<-4\pi$

$$w(r) < \frac{2}{\pi} \frac{r_1^2}{\rho^2} \quad \text{for } r>\rho$$

$$\longrightarrow 0$$

as $\beta \to -8\pi$ and $r_1 \to 0$. Thus, w labelled by β form a standard approximating sequence of delta functions as $\beta \to -8\pi$. In particular, for any smooth test function

$$\lim_{\beta\to -8\pi} \int_D \varphi(\mathbf{x})\, w_\beta(\mathbf{x})\, d^2\mathbf{x} = \varphi(0).$$

The energy is calculated as

$$E = \frac{1}{2} \int_D |u_\theta|^2 \, d^2x \qquad \text{since } u_r = 0$$

$$= \frac{1}{2} \int_0^1 2\pi r \, dr \left(\frac{4A}{\beta} \frac{r}{1 - Ar^2} \right)^2$$

$$= \frac{8\pi A}{\beta^2} \int_0^1 du \, \frac{Au}{(1 - Au)^2} \qquad \text{with } u = r^2$$

$$= \frac{8\pi A}{\beta^2} \int_0^1 du \left[\frac{-1}{1 - Au} + \frac{1}{(1 - Au)^2} \right]$$

$$= \frac{8\pi}{\beta^2} \left[\ln(1 - A) - \frac{A}{1 - A} \right]$$

$$\sim \frac{1}{8\pi} \ln|A| \qquad \text{as } \beta \to -8\pi, \; A \to -\infty$$

$$\sim -\frac{1}{4\pi} \ln r_1 \qquad \text{using } r_1 = \frac{1}{\sqrt{A}}$$

so that

$$r_1 \sim e^{-4\pi E}$$

at high energies.

(c) We see that

$$u_\theta = \frac{\partial \psi}{\partial r} = \begin{cases} \frac{1}{2}\omega_0 r & r < r_1 \\ \frac{1}{2}\omega_0 r_1^2 \cdot \frac{1}{r} & r > r_1 \end{cases}$$

and

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \begin{cases} \omega_0 & r < r_1 \\ 0 & r > r_1 \end{cases}$$

which is indeed a circular vortex patch. The normalization $\int_D \omega\, d^2x = 1$ requires that $\omega_0 = 1/\pi r_1^2$, so that

$$u_\theta = \begin{cases} \frac{1}{2\pi}\frac{r}{r_1^2} & r < r_1 \\ \frac{1}{2\pi}\frac{1}{r} & r > r_1 \end{cases}$$

The energy has two parts

$$E^< = \frac{1}{2}\int_0^{r_1} \frac{1}{(2\pi r_1^2)^2} r^2 \cdot 2\pi r\, dr = \frac{1}{4\pi r_1^4}\int_0^{r_1} r^3 dr = \frac{1}{16\pi}$$

and

$$E^> = \frac{1}{2}\frac{1}{(2\pi)^2}\int_{r_1}^{1} \frac{1}{r^2}\cdot 2\pi r\, dr = \frac{1}{4\pi}\int_{r_1}^{1}\frac{dr}{r} = -\frac{1}{4\pi}\ln r_1$$

The second dominates for small r_1, so that $E \sim -\frac{1}{4\pi}\ln r_1$.

Problem 5. (a) Two wavenumbers **p** and **q** that interact to transfer enstrophy or energy to wavenumber **k** must satisfy

$$\mathbf{k} = \mathbf{p} + \mathbf{q}$$

and thus

$$|\mathbf{k}| \le |\mathbf{p}| + |\mathbf{q}| < 2K$$

if $|\mathbf{p}|, |\mathbf{q}| < K$.

(b) The quantity $\eta = \nu \langle |\nabla \omega|^2 \rangle$ has dimension

$$[\eta] = \frac{(\text{length})^2}{\text{time}} \cdot \left(\frac{1}{\text{length} \cdot \text{time}} \right)^2$$

$$= \frac{1}{(\text{time})^3}$$

Hence, the turnover rate is determined dimensionally to be

$$T_k \cong \eta^{-1/3}$$

independent of wavenumber k. This is quite distinct from the 3D energy cascade which is "accelerated," so that turnover times decrease as a power of $\frac{1}{k}$ for $k \to \infty$.

(c) The number of steps required is $\log_2\left(\frac{k_2}{k_1}\right) = \frac{\ln(k_2/k_1)}{\ln 2}$ and the time required is

$$T \cong \eta^{-1/3} \ln\left(\frac{k_2}{k_1}\right).$$

Conversely, the wavenumber reached in time t will be

$$k_2(t) = 2^{t/\tau} \cdot k_1$$

$$= k_1 \exp\left(\frac{\ln 2}{\tau} t \right)$$

$$\cong k_1 \exp\left(\eta^{1/3} t \right).$$

(d) In the k^{-3} range

$$\frac{1}{2} \omega_{rms}^2 = \int_{k_1}^{k_2} \eta^{2/3} \frac{dk}{k} \cong \eta^{2/3} \ln\left(\frac{k_2}{k_1}\right)$$

so that

$$\omega_{rms} \cong \eta^{1/3} \left[\ln\left(\frac{k_2}{k_1}\right) \right]^{1/2}$$

(e) Using (c) and (d),

$$T \cong \frac{1}{\omega_{rms}} \left[\ln\left(\frac{k_2}{k_1}\right) \right]^{3/2}.$$

For the synoptic scales of Earth's atmosphere

$$T \cong 10^4 \text{ sec } (\ln 4)^{3/2} \cong 1.63 \times 10^4 \text{ sec}$$

$$\cong 4.53 \text{ hr}$$