Homework No.1, 550.696, Due February 14, 2014.

1. (a) The *Poincaré-Sobolev wave operator* $\mathcal{L}$ for rotating incompressible fluids is defined by

$$\mathcal{L}\mathbf{u} = -\hat{\mathbf{z}}\times\mathbf{u} + \boldsymbol{\nabla} q$$

for $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and with q determined so that also $\boldsymbol{\nabla}\cdot(\mathcal{L}\mathbf{u}) = 0$. Show that

$$\boldsymbol{\nabla}\times(\mathcal{L}\mathbf{u}) = \mathcal{L}(\boldsymbol{\nabla}\times\mathbf{u}).$$

(b) The *helical modes* with wavenumber $\mathbf{k}$ are defined (up to normalization) by

$$\mathbf{h}_s(\mathbf{k}) = \hat{\mathbf{z}}^{\perp} + is\ \hat{\mathbf{k}}\times\hat{\mathbf{z}}, \quad s = \pm 1$$

with $\hat{\mathbf{z}}^{\perp} = \hat{\mathbf{z}} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$, so that $\mathbf{k}\cdot h_s(\mathbf{k}) = 0$. Show also that

$$i\mathbf{k}\times h_s(\mathbf{k}) = s|\mathbf{k}|h_s(\mathbf{k}), \quad \mathcal{L}h_s(\mathbf{k}) = is(\hat{\mathbf{k}}\cdot\hat{\mathbf{z}})h_s(\mathbf{k})$$

where $\mathcal{L}\hat{\mathbf{u}}(\mathbf{k}) = -\hat{\mathbf{z}}\times\hat{\mathbf{u}}(\mathbf{k}) + i\mathbf{k}\hat{q}(\mathbf{k})$ with $\hat{q}$ determined so that $\mathbf{k}\cdot\mathcal{L}\hat{\mathbf{u}}(\mathbf{k}) = 0$.

2. (a) Previously in the course we have defined the total kinetic energy at length scales less than ℓ by

$$E_{<\ell}(t) = \frac{1}{2}\int_D d^dx\ \tau_\ell(u_i, u_i).$$

On the 2D torus $D = [0, 2\pi]^2$ we must use filter kernels G_ℓ^P that are periodizations of filter kernels G on $\mathbb{R}^2$ defined by

$$G_\ell^P(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} G_\ell(\mathbf{r} + 2\pi\mathbf{n}).$$

Show that *Fjørtoft's energy bound* holds for $E_{<\ell}(t)$, in the form

$$E_{<\ell}(t) \le C\ell^2\Omega(t)$$

for a fixed constant C, if the Fourier transform of G satisfies $|\hat{G}(\mathbf{k})|^2 = 1 - O(|\mathbf{k}|^2)$. *Hint:* Use the Poisson summation formula.

(b) Likewise, show that *Fjørtoft's enstrophy bound* holds on the 2D torus $D = \mathbb{T}^2$ for total enstrophy in scales greater than ℓ

$$\Omega_{>\ell}(t) = \frac{1}{2}\int_D d^2x\ |\bar{\omega}_\ell|^2,$$

in the form

$$\Omega_{>\ell}(t) \le C\frac{1}{\ell^2}E(t)$$

with a fixed constant C for any filter kernel with $\boldsymbol{\nabla} G \in L^1(\mathbb{R}^2)$.

3. (a) In 3D incompressible fluids, the helicity

$$H(t) = \int d^3x \; \boldsymbol{\omega}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t)$$

is an inviscid invariant, in addition to kinetic energy. By the Cauchy-Schwartz inequality $|\hat{\boldsymbol{\omega}}(\mathbf{k},t)\cdot\hat{\mathbf{u}}(\mathbf{k},t)| \leq |\mathbf{k}||\hat{\mathbf{u}}(\mathbf{k},t)|^2$, so that helicity and energy spectra satisfy the inequality

$$|H(k,t)| \leq 2kE(k,t).$$

Does this bound imply by Fjørtoft's argument that energy in decaying 3D turbulence must remain in large scales, as in 2D? Must helicity remain in small scales?

(b) Now consider a modified Navier-Stokes dynamics keeping only positive helicity modes. This can be defined in the helical basis as

$$(\partial_t - i\omega_+(\mathbf{k}) - \nu k^2)a_+(\mathbf{k},t) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} C^{+++}_{\mathbf{kpq}} a^*_+(\mathbf{p},t)a^*_+(\mathbf{q},t).$$

Repeat part (a) for this modified dynamics.

4. Consider the stream function in polar coordinates (r,θ),

$$\psi(r) = -\frac{2}{\beta}\ln(1-Ar^2), \quad A = \frac{\beta}{8\pi+\beta}, \quad \beta > -8\pi$$

inside the two-dimensional unit disk $D = \{\mathbf{x}\in\mathbb{R}^2 : |\mathbf{x}| \leq 1\}$.

(a) Show that

$$Z \equiv \int_0^1 2\pi r dr \; e^{\beta\psi(r)} = \frac{\pi}{1-A}$$

and

$$\omega(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{1-A}{\pi}\frac{1}{(1-Ar^2)^2},$$

giving a solution of the equation $\triangle\psi = \frac{1}{Z}e^{\beta\psi}$ in Onsager's equilibrium vortex theory.

(b) Show that in the limit $\beta \to -8\pi$, the vorticity field becomes a delta function at the origin and the energy $E \sim -\ln(r_1)/4\pi$, with $r_1 = |A|^{-1/2}$ the effective radius.

(c) The latter result could be guessed from the *circular vortex patch* with vorticity level $\omega_0 = 1/\pi r_1^2$ and radius $r_1 < 1$, whose stream function in the disk is given by

$$\psi(\mathbf{x}) = \begin{cases} \frac{1}{4}\omega_0 r^2 & r < r_1 \\ \frac{1}{4}\omega_0 r_1^2\left(1 + 2\ln\left(\frac{r}{r_1}\right)\right) & r > r_1 \end{cases}$$

Show here also that $E \sim -\ln(r_1)/4\pi$ for $r_1 \ll 1$.

5. Here we explore a cascade interpretation of *Batchelor's* k^{-3} *range.*

(a) Explain why two vortical modes with wavenumber magnitudes $< K$ can transfer their vorticity to a mode with wavenumber magnitude at most $2K$.

(b) Use a K1-style dimensional argument to estimate the turnover time of eddies at wavenumber k in Batchelor's k^{-3} range in terms of entropy dissipation rate η.

(c) If the enstrophy is transferred to small scales in a stepwise cascade with wavenumbers increasing by a factor of 2, how many steps and how long will it take for transfer from wavenumber k_1 to wavenumber $k_2 > k_1$? Conversely, what wavenumber k_2 will be reached in time t starting from wavenumber k_1?

(d) Find a relationship between the rms vorticity level ω_{rms} and the enstrophy dissipation rate η for a Batchelor k^{-3} range over wavenumber interval $k_1 < k < k_2$.

(e) A k^{-3} energy spectrum is observed in the Earth's atmosphere over about two octaves of wavenumber, for lengths 500 km - 2000 km (synoptic scales). Typical vorticity levels in the atmosphere are $\omega_{rms} \sim 10^{-4}$ sec^{-1}. Estimate the total time required for enstrophy cascade across this two-octave range.