

Research



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Unsteady separation in vortex-induced boundary layers

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This paper provides a brief review of the analytical and numerical developments related to unsteady boundary-layer separation, in particular as it relates to vortex-induced flows, leading up to our present understanding of this important feature in high-Reynolds-number, surface-bounded flows in the presence of an adverse pressure gradient. In large part, vortex-induced separation has been the catalyst for pulling together the theory, numerics and applications of unsteady separation. Particular attention is given to the role that Prof. Frank T. Smith, FRS, has played in these developments over the course of the past 35 years. The following points will be emphasized: (i) unsteady separation plays a pivotal role in a wide variety of high-Reynolds-number flows, (ii) asymptotic methods have been instrumental in elucidating the physics of both steady and unsteady separation, (iii) Frank T. Smith has served as a catalyst in the application of asymptotic methods to high-Reynolds-number flows, and (iv) there is still much work to do in articulating a complete theoretical understanding of unsteady boundary-layer separation.

1. Introduction and background

Although the viscosity of many fluids of interest is very small, its effects can have disproportionate qualitative, not just quantitative, effects. This realization became evident when, in 1904, Ludwig Prandtl elucidated the idea of the *boundary layer*, a thin region near solid surfaces in which the velocity is adjusted from zero owing to the no-slip condition at the surface to the $O(1)$ velocity

corresponding to the inviscid (frictionless) flow past the object. Even though the viscosity is small, therefore, the large gradients in velocity across the boundary layer allow for the streamwise viscous forces to be of the same order as the inertial (convective) terms in the Navier–Stokes equations. In other words, the inviscid Euler equations, while governing in the majority of high-Reynolds-number, Re , external flows, are not uniformly valid, and viscous effects must be accounted for in the thin boundary layers. This seemingly simple idea gave birth to a whole new approach to analysing fluid flows. In particular, although Prandtl’s original work was rather heuristic, it led to the development of asymptotic methods that formalize the analysis of systems governed by differential equations having small (or large) parameters.

Prandtl’s classical boundary-layer equations, which can be formally derived in the limit as $Re \rightarrow \infty$, depend on the assumption that the boundary layer remains sufficiently thin and has negligible influence on the outer inviscid flow. This is the case for high-Reynolds-number attached flows, where the fluid passes smoothly over solid objects, such as a flat plate. However, as Prandtl himself recognized, this picture breaks down in the presence of an adverse pressure gradient acting in the streamwise direction on a boundary layer, i.e. such that the pressure is increasing in the flow direction. This causes the low-momentum fluid adjacent to a surface to become susceptible to being forced upstream against the balance of the flow, leading to separation. A pronounced separation region can then lead to the boundary layer no longer remaining thin along the surface.

Goldstein [1] found that such an occurrence is catastrophic in the context of a steady boundary layer in the sense that the classical steady boundary-layer equations become singular at the separation point where the wall shear stress vanishes. The singularity arises as the velocity normal to the surface at the outer edge of the boundary layer increases without bound, thereby violating the assumption that the boundary layer is thin. As a result of the Goldstein singularity, numerical calculations of the steady boundary-layer equations with prescribed pressure gradient over a fixed wall are terminated at the point of zero wall shear. Further integration of the steady boundary-layer equations into the reversed flow region is then impossible.

Work began on developing a theory that would determine the necessary physics that must be incorporated in order to avoid the Goldstein singularity. Based on the seminal ideas of Lighthill [2,3] on self-induced separation, *triple-deck theory* was formalized in the late 1960s for incompressible flow [4,5] and supersonic flow [6]. These original papers on triple-deck theory focused on the flow near the trailing edge of a flat plate. The triple-deck structure was first developed by Prof. Keith Stewartson when he visited The Ohio State University in the summer and autumn of 1968 at the request of Prof. Odus Burggraf. Legend has it that he named his theory at a local Columbus *Burger King* adjacent to the Ohio State campus while eating a ‘triple deck’ hamburger. Stewartson would return to Ohio State on multiple occasions, as did Prof. Frank T. Smith many years later. It was later shown by Smith [7,8] that this same triple-deck structure avoids the Goldstein singularity at a separation point in incompressible flow. Beyond the boundary layer itself, it is not an overstatement to assert that triple-deck theory is the single most successful asymptotic framework in fluid mechanics. It is remarkable for its generality as well as its relative simplicity. The essential realization inherent in triple-deck theory is that the interaction between the viscous boundary layer and outer inviscid flow must be handled in a truly coupled fashion for flows involving separation.

The encouraging results obtained using triple-deck theory led to a golden age of asymptotics in fluid mechanics beginning in the 1970s as the widespread applicability of triple-deck theory was realized and applied mathematicians were emboldened to seek out other asymptotic solutions (e.g. [9]). Triple-deck theory was found to apply in a wide variety of geometries, flow situations and across the full spectrum of flow regimes, including subsonic, transonic, supersonic and hypersonic. Frank T. Smith was one of the strongest proponents of triple-deck theory as he extended its range of applications throughout the 1970s in a series of papers that focus on steady flows. His review article ‘On the high Reynolds number theory of laminar flows’ [8] succinctly captures and categorizes the results for steady boundary-layer separation and hints at the breakthroughs in unsteady separation to come.

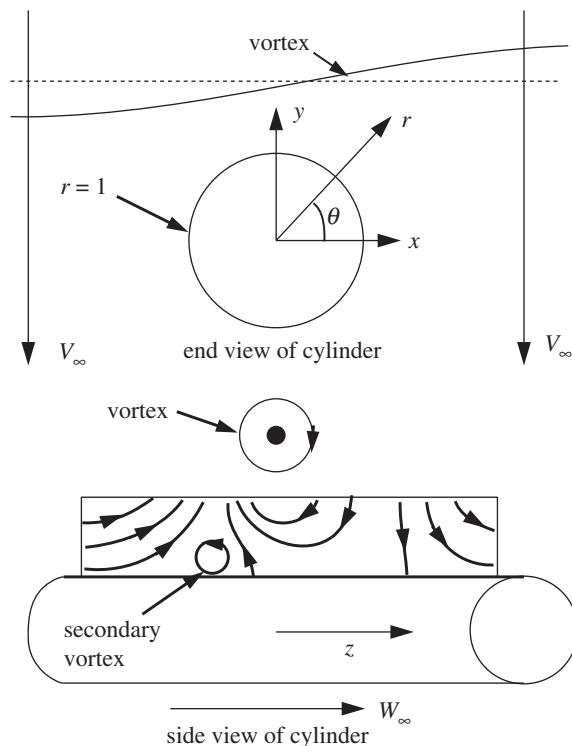


Figure 1. Schematic of a three-dimensional vortex filament and its interaction with a circular cylinder. The vortex filament represents the tip-vortex shed from a helicopter rotor blade and the cylinder represents a simplified model of a helicopter airframe (from [13]).

With the success of triple-deck theory in elucidating steady separation, and owing to the prevalence of unsteady separation in many important applications, researchers began turning their attention to unsteady separation in earnest. However, it was soon found that the unsteady case was far more difficult to understand.

Both steady and unsteady boundary-layer separation are induced by an adverse streamwise pressure gradient acting on a boundary layer. This adverse pressure gradient may result from the geometry, such as in the flow around a circular cylinder or the leading-edge of an aerofoil at angle of attack, or owing to the presence of a vortex adjacent to a solid surface. The latter case is our focus in this brief review. Such a vortex may occur in the boundary layer approaching a surface-mounted obstacle, i.e. a juncture flow, vortices shed from upstream surfaces, coherent vortex structures occurring in turbulent boundary layers, and branching pipes and vessels [10]. A particularly interesting application that brings together a number of these effects is the interaction of a rotor-tip vortex with a helicopter airframe. A simplified depiction of this latter problem is provided in figure 1. The complicated unsteady and three-dimensional vortex wake is responsible for many peculiar aerodynamic features of a helicopter in flight [11–13]. First, the tip-vortex will interact with the rotor blades, a phenomenon referred to as *blade/vortex interaction* (BVI). This interaction causes vibration of the rotor blades and noise radiated to the far field. BVI noise is responsible for the familiar sound of an approaching helicopter. Second, the vortical wake interacts with the fuselage of the vehicle (figure 1), producing a large-amplitude short-scale pressure load on the airframe. Third, the main rotor wake interacts with the tail rotor and can greatly affect the handling qualities of the helicopter. See the reviews by Conlisk [14,15] for details of helicopter aerodynamics.

As a means of honouring Prof. Frank T. Smith, FRS, hereafter referred to (affectionately) as FTS, we review his essential research primarily related to separation theory in the context

of boundary-layer separation induced by vortices in incompressible flows. This is only one avenue for which his extensive research on separated flows is relevant, but it will serve to focus the discussion of his contributions in this important area. In addition, it is an area for which the authors have collaborated with FTS and/or made complementary contributions. The paper also serves as a review of this important field and can be considered to be an update of the excellent review by Doligalski *et al.* [10]. Subsequent sections recount the developments in the asymptotic theory of unsteady separation, stability of separating flows, and recent work on better understanding unsteady separation through solutions of the full Navier–Stokes equations. The last includes both the onset of separation and instability, thereby bringing us to our contemporary understanding of unsteady separation. The paper closes with a summary and discussion highlighting the need for future work in this area.

2. Asymptotic theory of unsteady separation

With a relatively complete picture of steady separation in place and emboldened by the success of triple-deck theory in avoiding the Goldstein singularity, FTS and others embarked on a two-decade journey to revolutionize our understanding of unsteady separation (among other pursuits). This is also when his research took a decidedly collaborative turn. Throughout the 1970s, the vast majority of his papers were single-author expositions that served to set the tone and provide examples of the application of rigorous asymptotic methods in fluid mechanics. Beginning in the mid-1980s, many of his papers were co-authored with researchers from a broad range of institutions throughout the UK and USA. With his guidance and leadership, these collaborations served to accelerate developments in a number of fields.

(a) Classical non-interactive boundary layers

Led by our understanding of steady separation, early efforts in studying unsteady boundary-layer separation were made by tracking the point of zero wall shear. Between 1956 and 1958, Moore [16], Rott [17] and Sears [18] worked independently on the problem of steady boundary-layer flow past a moving wall. They all came to the same conclusion that, in such a case, the vanishing wall shear and the accompanying flow reversal near the wall do not necessarily denote separation in the physical sense of flow breaking away from the wall. In choosing the reference frame in which the wall is at rest, the flow is unsteady. This observation led a number of investigators to search for more generalized criteria of separation that would hold for both steady and unsteady flows [19].

Moore [16] proposed that separation occurs at the point of zero shear stress (vorticity) occurring *within* the boundary layer, rather than at the surface as in the steady case, at a location where the streamwise velocity vanishes in a frame of reference moving with this separation point. This has become known as the Moore–Rott–Sears (MRS) criterion, as it was also arrived at independently by Rott and Sears. Because the steady flow over a moving wall is analogous to a special case of unsteady flow over a fixed wall, the development of a singular boundary-layer solution could also be an indication of separation for an unsteady flow [20,21]. The singularity would again show up as a drastic thickening of the boundary layer. This argument, together with the MRS criterion, led to an important change in concept, at least in the following senses. First, zero wall shear does not necessarily correspond to separation in unsteady flow. There may not be any sign of drastic boundary-layer thickening at such a point. In other words, the boundary-layer approximation may remain valid beyond the point of zero wall shear to include a reversed-flow region. Second, the emergence of a singularity signals the breakdown of the boundary-layer approximation. This singular behaviour occurs as the boundary layer penetrates further into the external flow than allowed for in the boundary-layer approximation. It also means that some of the terms in the Navier–Stokes equations neglected in the boundary-layer approximation would have to become important to keep the separating motion finite.

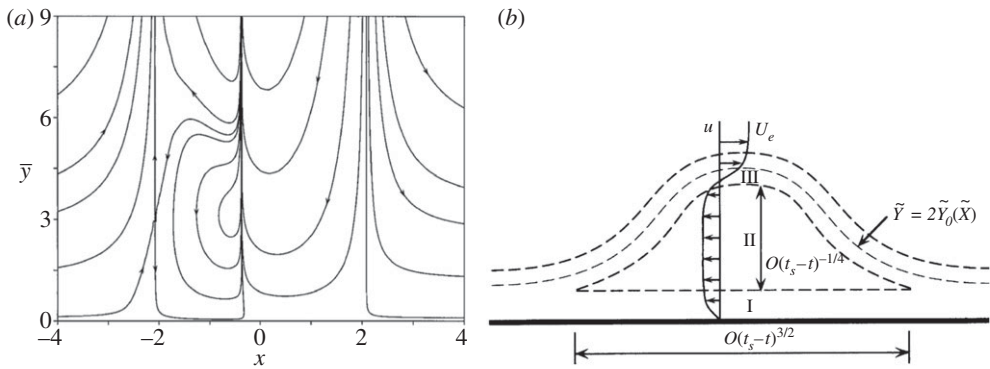


Figure 2. The Van Dommelen and Shen singularity in non-interactive, unsteady boundary-layer solutions. (a) Streamlines at $t_s = 1.402$ from the non-interactive boundary-layer solutions for the thick-core vortex (from [32]). (b) Schematic of the terminal boundary-layer structure within the Van Dommelen and Shen singularity (from [33]).

Throughout the 1970s, several unsuccessful attempts were made to numerically confirm the presence of a singularity in the unsteady boundary-layer equations according to the MRS criterion (e.g. [22–27]). All of these studies used the traditional Eulerian formulation of the unsteady boundary-layer equations.

The nature of the difficulties associated with unsteady boundary-layer separation was finally elucidated by Van Dommelen & Shen [28], who calculated the boundary layer on an impulsively started circular cylinder using an unconventional technique based on the Lagrangian formulation of the unsteady boundary-layer equations. In Lagrangian coordinates, the boundary-layer equations give the position (x, y) at time t of a particle whose position is (ξ, η) at time $t=0$. Van Dommelen & Shen [28] and Van Dommelen [29] have shown that the particle position x is non-singular, but a singularity occurs when x has a stationary point, i.e. $\partial x / \partial \xi = \partial x / \partial \eta = 0$. This condition implies that two particles that were an infinitesimal distance $\partial \xi$ apart have reached the same location x . When transformed back to the Eulerian coordinate, $\partial u / \partial x = (\partial u / \partial \xi) / (\partial x / \partial \xi) \rightarrow \infty$ at the singular point. As pointed out by Van Dommelen [29], while so simple, this expression is very significant because it explains why it is so difficult to integrate the boundary-layer equations in Eulerian coordinates. By careful numerical integration in Lagrangian coordinates, Van Dommelen & Shen [28] finally located the singular point for the impulsively started circular cylinder at 111° from the forward stagnation point and at time $t_s = 1.5025$, i.e. after the cylinder moves from its initial position about 1.5 radii.

The physical separation process was explained by Van Dommelen & Shen [28,30]. According to the MRS condition, the separation point must occur on the zero vorticity line bisecting a recirculation region. When a stationary point develops on this zero-vorticity line, it means fluid elements moving on this zero-vorticity line have arrived at the same location. Mass conservation then forces the fluid elements to expand rapidly in the direction normal to the wall, thereby marking the onset of the Van Dommelen and Shen singularity [29].

The reason it took so long to fully appreciate the physical and mathematical consequences of such an occurrence is that it is inherently difficult to identify such a moving separation point within the typical Eulerian context. By contrast, the Lagrangian formulation of the unsteady boundary-layer equations has the following two advantages: (i) it decouples the motion in the streamwise direction from that in the normal direction; it is the normal velocity and position that become singular in the terminal boundary-layer solution; and (ii) there is a clear and definitive criterion for a singularity in the boundary-layer equations.

Peridier *et al.* [31] carried out a Lagrangian calculation of the same vortex-driven boundary layer on a plane wall as considered in Walker [26] and Doligalski & Walker [27]. For example, see figure 2a for the singular boundary-layer solution arising for the thick-core vortex. Their

conclusions agree with that of Van Dommelen and Shen. In conventional Eulerian coordinates, the evidence for a singularity in the classical unsteady boundary-layer formulation also was strongly supported by the adaptive-grid calculations of Adams *et al.* [34], which were in excellent agreement with the results of Peridier *et al.* [31] for the problem of a vortex convecting above a plane wall. These computational results are for two-dimensional problems. In three dimensions, however, the work is much more limited. It includes the Lagrangian description of unsteady boundary-layer separation in three dimensions by Van Dommelen & Cowley [35] and numerical calculation in an Eulerian framework performed by Wu & Shen [36] and Affes *et al.* [13] in the context of the interaction of a tip-vortex with a cylinder simulating a helicopter airframe. They all found that a singularity emerges and displays essentially the same nature as in two dimensions.

In the infinite-Reynolds-number limit, the unsteady, non-interactive boundary layer terminates in the Van Dommelen and Shen singularity in finite time, i.e. the spike in figure 2*a*. The theoretical structure of this singularity was determined by Van Dommelen & Shen [30] (see also [37]) and is called the *terminal boundary-layer solution*. It consists of the viscous boundary layer bifurcated into two viscous layers above and below a vorticity-depleted region in which the velocity is nearly constant, as shown schematically in figure 2*b*. The terminal solution is a similarity solution for the growing spike in the boundary layer with the height and width of the central region expanding and contracting like $O(t_s - t)^{-1/4}$ and $O(t_s - t)^{3/2}$ as $t \rightarrow t_s$, respectively, where t_s is the time at which the non-interactive Van Dommelen and Shen singularity occurs. On the boundary-layer scale, therefore, the terminal solution collapses to zero width in the streamwise direction and infinite thickness in the normal direction, thereby ejecting boundary-layer vorticity away from the surface with a maximum displacement velocity that is $O(t_s - t)^{-7/4}$. Strikingly, the terminal boundary-layer structure is independent of the local pressure gradient distribution that initiated the separation process in the first place; therefore, it is generic and provides a common gateway to the subsequent stages regardless of the overall setting that produced the adverse pressure gradient.

Summarizing, the common features in high-Reynolds-number unsteady separation include formation of a growing recirculation region along a surface owing to an adverse streamwise pressure gradient, rapid focusing of the flow on the upstream side of the recirculation region, followed by a sudden small-scale eruption of the boundary layer, leading to the Van Dommelen and Shen singularity during which the classical non-interactive boundary-layer equations are no longer valid. The hallmark of unsteady separation, therefore, is the sudden eruption of near-wall vorticity away from the surface that penetrates the outer flow.

(b) Interactive boundary layers

Despite their differences, triple-deck theory in the steady case suggested that viscous–inviscid interaction may be the missing physics that leads to the non-interactive Van Dommelen and Shen singularity. Like the strong outflows that occur owing to the Goldstein singularity in steady separation, strong outflows also occur locally in the region of the Van Dommelen and Shen singularity in the unsteady scenario. There are two complementary approaches for incorporating viscous–inviscid interaction effects, which are consistent in the limit as $Re \rightarrow \infty$. First, *interactive boundary-layer theory* considers large Reynolds numbers and solves for the coupled viscous boundary layer and the streamwise pressure gradient owing to the outer inviscid flow. In the interacting boundary-layer formulation, the influence of the viscous boundary-layer flow on the external potential flow is imposed by the introduction of a correction to the inviscid solution along the surface.

The singularity in the interacting boundary-layer flow was first suggested in the work of Brotherton-Ratcliffe & Smith [38], who studied the problem of interacting boundary-layer flow over a surface distortion and a liquid layer. Their results show that, when the boundary-layer flow is coupled with the outer inviscid flow, the boundary layer is susceptible to a rapidly growing viscous–inviscid instability. This instability promotes a complete nonlinear breakdown of the interacting boundary layer within a finite scaled time. Smith [39] generalized the work of

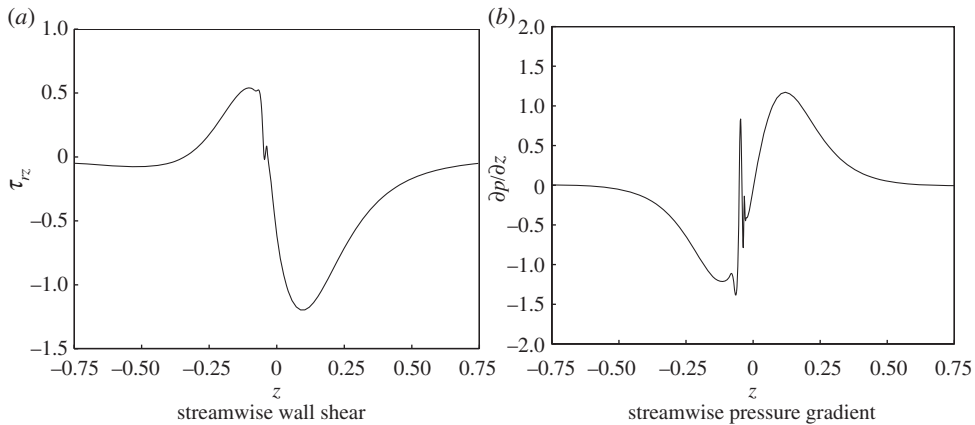


Figure 3. Results on the symmetry plane ($x = 0$ on the top of the airframe in figure 1) for $Re = 10^7$ at $t = 0.73$ (from [42]). (a) Streamwise wall shear, (b) streamwise pressure gradient.

Brotherton-Ratcliffe & Smith [38] and concluded that a singularity can occur at finite time in any unsteady interacting boundary-layer formulation. It was suggested that the streamwise pressure gradient approaches a singularity of the form

$$\left. \frac{dp}{dz} \right|_{\max} \sim O(t_s - t)^{-1} \quad \text{as } t \rightarrow t_s. \quad (2.1)$$

Moreover, applying the z -momentum equation at the wall, Peridier *et al.* [40] showed that the local shear stress distribution on the wall will also develop a singularity according to

$$\tau_{rz}|_{\max} \sim O(t_s - t)^{-1/4} \quad \text{as } t \rightarrow t_s. \quad (2.2)$$

The presence of the interactive singularity was suggested by Chuang & Conlisk [41] and formally confirmed by Peridier *et al.* [40]. They found that the Smith [39] singularity emerges for Reynolds numbers ranging from 10^5 to 10^8 . The predictions were also quantitatively confirmed in the calculations of three-dimensional unsteady interacting boundary-layer flow using a uniformly distributed grid by Xiao *et al.* [42]. Figure 3 shows the distribution of wall shear and pressure gradient on the symmetry plane at $t = 0.73$ for $Re = 10^7$.

The second approach for incorporating viscous–inviscid interaction is to continue the formal asymptotic expansions in the limit as $Re \rightarrow \infty$. In this limit, recall that the unsteady non-interactive boundary layer terminates in the Van Dommelen and Shen singularity in finite time as discussed in the previous section. Before becoming singular at t_s , the growing boundary layer provokes an interaction with the outer inviscid flow. This is called the *first interactive stage* and was formulated by Elliott *et al.* [37]. This stage occurs on the time scale $O(Re^{-2/11})$ prior to the non-interactive singularity. As shown schematically in figure 4, an asymptotic structure is set up wherein the $O(Re^{-1/2})$ bifurcated boundary layer surrounds the $O(Re^{-5/11})$ by $O(Re^{-3/11})$ vorticity-depleted region. To accommodate interaction with the outer inviscid flow, an $O(Re^{-3/11})$ by $O(Re^{-3/11})$ region immediately above the growing spike is coupled through an interaction condition with the boundary layer. The first interactive stage was predicted to terminate in another singularity according to Smith [39] at time t_{sl} . Unfortunately, the interactive singularity was found to occur at an earlier time than the non-interactive singularity, i.e. $t_{sl} < t_s$ [40].

In order to relieve the interactive singularity, it was determined that normal pressure gradient effects come into effect within the growing boundary layer. Recall that, in the non-interactive boundary-layer context (as well as the first interactive stage), the normal pressure gradients vanish to leading order as $Re \rightarrow \infty$. This then is the first stage during which normal pressure gradients become $O(1)$. This stage was formulated by Holye *et al.* [43] and considered in more

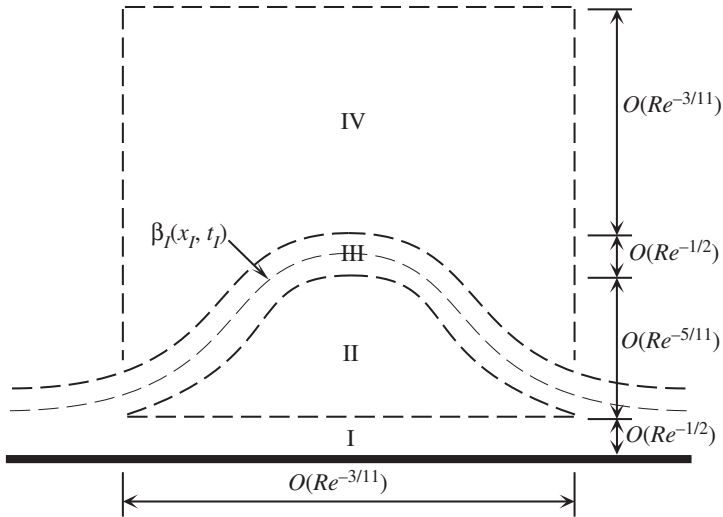


Figure 4. Schematic of the first interactive stage of unsteady separation (from [33]).

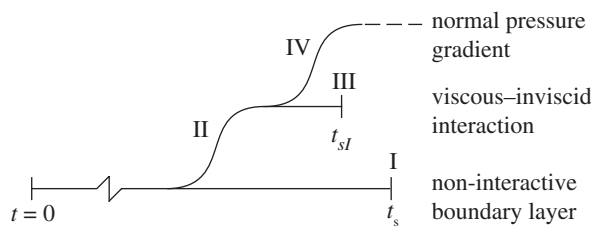


Figure 5. Schematic of the initial asymptotic stages of unsteady separation (from [47]).

detail by Li *et al.* [44] and Smith *et al.* [45] (see also [46]). Using the interactive boundary-layer equations as the starting point, normal pressure gradient effects are found to arise owing to a critical layer near an inflection point in the streamwise velocity profiles. The first three asymptotic stages of unsteady separation are summarized in figure 5. Time moves from left to right in the schematic, and each tier represents a unique asymptotic stage governed by a subset of the full Navier–Stokes equations as described earlier.

3. Stability of separating boundary layers

One of the great difficulties in dealing with high-Reynolds-number flows is their susceptibility not only to separation but also to instability. Although the vast majority of numerical solutions of the unsteady non-interactive boundary-layer equations have not exhibited any signs of instability, stability analyses do reveal the possibility of various types of instabilities. In particular, a class of instabilities may occur once a zero shear stress point appears within the boundary layer; recall that this is also a necessary condition for the onset of the non-interactive Van Dommelen and Shen singularity according to the MRS criterion. The case in which the point of zero shear stress forms adjacent to the surface at a velocity minimum in the context of marginal separation was considered by Smith & Elliott [48]. Cowley *et al.* [49] consider the scenario for which the zero shear stress point occurs away from the wall, and Bhaskaran *et al.* [50] consider the transition between these two cases. The maximum growth rate of this class of instabilities is only $O(k^{1/2})$, where k is the wavenumber; therefore, disturbances may not have sufficient time to grow prior

to the onset of the singularity at t_s , which is consistent with the numerical results that were able to calculate the unsteady boundary-layer equations using Lagrangian coordinates through to the time of the singularity without any signs of an instability. Note that it is possible that this is due to the lack of sufficient resolution to identify the instability.

In general, the type of instability that may arise within the boundary-layer (and triple-deck) context depends upon the streamwise length scale of the disturbances (with wavenumber k) relative to the normal thickness of the corresponding viscous boundary layer. In the case of the $O(1)$ by $O(Re^{-1/2})$ boundary layer, therefore, the Cowley–Hocking–Tutty instability may arise when $O(1) \gg 1/k \gg O(Re^{-1/2})$. Although additional physics is necessary in order for it to show up in boundary-layer solutions, there is also the possibility of an inviscid Rayleigh instability, in which the streamwise disturbances are of the same length scale as the boundary-layer thickness, i.e. $1/k \sim O(Re^{-1/2})$. This possibility will be addressed in the next section.

Let us return to the first interactive stage and describe the results of Cassel *et al.* [33] and their implications. Following the success of the application of the Lagrangian formulation in calculating the unsteady boundary-layer equations through to the non-interactive Van Dommelen and Shen singularity, a Lagrangian calculation of the first interactive stage was pursued. The initial condition is the terminal boundary-layer solution for the non-interactive boundary layer, and the inviscid equations are solved within the vorticity-depleted region that forms between the bifurcated viscous layers subjected to the incompressible interaction condition. Based on the interactive boundary-layer calculations of Peridier *et al.* [40], a singularity was expected to occur earlier than the time at which the Van Dommelen and Shen singularity would occur if interaction was not included. Unexpectedly, the numerical results also revealed the onset of a high-frequency instability at the very onset of interaction. That is, the terminal solution is unstable once the interaction is ‘turned on’. This instability was confirmed through a linear stability analysis of the first interactive stage using velocity profiles from the initial condition (terminal boundary-layer solution). These results called into question the validity of the first interactive stage as well as serving as a reminder that high-Reynolds-number flows are susceptible to instabilities.

4. Navier–Stokes solutions of unsteady separation

Pursuit of an asymptotic structure of unsteady separation has led to a number of unanswered questions, in particular in the light of the instabilities that have been shown to be possible, or in fact have been shown to appear, in unsteady separating flows. The most troubling finding is that the terminal boundary-layer solution is found to be unstable at the very onset of interaction during the first interactive stage. In an effort to shed light on these issues and to substantiate the first three asymptotic stages of unsteady separation as described earlier, a long-term effort commenced to perform high-resolution, carefully formulated calculations of the full Navier–Stokes equations, for which it has been thought that no singularities occur. Unfortunately, such calculations must necessarily contend with two of the most challenging phenomena in fluid mechanics—separation and instability.

(a) Comparison with asymptotic stages

During the period of time in which the asymptotic theory of unsteady separation was being developed, computational solutions of the full Navier–Stokes equations could not be performed at sufficiently high Reynolds number in order to truly compare with or guide theoretical developments. In particular, these early studies were not at sufficiently high Reynolds number in order to observe the sharp spike in unsteady separation. One of the best examples of such a calculation is Koumoutsakos & Leonard [51], who obtained detailed results for the flow about an impulsively started cylinder for Reynolds numbers up to 9500. While these results hinted qualitatively at some of the basic features of unsteady separation from solid surfaces suggested by the theory, no quantitative comparisons were carried out. For the most part, computational

fluid dynamics (CFD) simulations had been (and still are) carried out at Reynolds numbers that are too low to capture the important features of the asymptotic theory. This is generally because researchers and practitioners are focused on larger scale flows in which unsteady separation may be only one of several important and spatially distinct features that need to be resolved. In order to overcome this limitation, and to focus exclusively on the unsteady separation event, Cassel [32] and Obabko & Cassel [47] performed highly resolved, high-Reynolds-number simulations for the boundary layer induced by a thick-core vortex above an infinite plane wall, in which the vorticity is distributed throughout the vortex (cf. the rectilinear vortex considered in [31,40] in which all of the vorticity is focused in the core).

For the thick-core vortex, unsteady non-interactive boundary-layer solutions reveal a recirculation region forming at approximately $t = 0.4$, a spike appearing on the upstream side of the recirculation region at $t = 1.3$, which terminates in the Van Dommelen & Shen singularity at $t_s = 1.402$ as shown in figure 2a [32]. In the corresponding Navier–Stokes solutions for the same problem with $Re = 10^5$, the recirculation region forms at approximately the same time, and a small-scale spike begins to form on the upstream side of the recirculation region at approximately $t = 1.1$. Recall that this is the earliest time at which the viscous–inviscid interaction would be expected to occur in the context of the infinite-Reynolds-number asymptotic theory. However, through analysis of the evolving streamwise pressure gradient along the wall, interaction is found to begin at approximately $t = 0.8$ for this Reynolds number. Such an interaction marks the first deviation between the boundary-layer and Navier–Stokes solutions. Because this interaction occurs well before formation of the spike and on a larger streamwise length scale, it has been referred to as *large-scale interaction* in contrast to the *small-scale interaction* induced by the growing spike in the limiting case. After spike formation, normal pressure gradients begin to grow dramatically in a very localized region just above the growing spike.

A more detailed investigation was carried out by Obabko & Cassel [47] for the same thick-core vortex model problem, and it was determined that, in terms of viscous–inviscid interaction, unsteady separation can be classified into three Reynolds number regimes, as summarized in figure 6. In the low-Reynolds-number regime, e.g. $Re = 10^3$, only large-scale interaction occurs owing to the strong outflows that develop on the upstream side of the growing recirculation region. In the moderate-Reynolds-number regime, e.g. $Re = 10^5$, large-scale interaction occurs, and a spike forms on the upstream side of the recirculation region that leads to a small-scale interaction. In the high-Reynolds-number regime, e.g. $Re \rightarrow \infty$, only small-scale interaction occurs owing to the growing spike. The effect of the large-scale interaction diminishes with increasing Reynolds number owing to the decreasing displacement thickness, i.e. $\delta = O(Re^{-1/2})$, while the influence of small-scale interaction diminishes with decreasing Reynolds number as the spike becomes wider and has a slower growth rate. Although the transition from the low- to moderate-Reynolds-number regime has been clearly identified, this is not the case for the moderate- to high-Reynolds-number transition. See figure 7 for results of the thick-core vortex illustrating the low- and moderate-Reynolds-number regimes. Observe that no spike forms in the low-Reynolds-number regime (figure 7a), whereas it does in the moderate-Reynolds-number regime (figure 7b,c). In summary, the overall asymptotic sequence has been confirmed within Navier–Stokes solutions for the thick-core vortex above a wall for large Reynolds numbers; however, a large-scale interaction is found to occur well before formation of the spike. Among other consequences, this large-scale interaction is responsible for accelerated spike formation, splitting of the primary recirculation region into co-rotating eddies due to local streamwise expansion, and a series of secondary recirculation regions leading to ejection of near-wall vorticity. It is important to note that the large-scale interaction is not simply a finite-Reynolds-number modification of the boundary-layer results for $Re \rightarrow \infty$; it occurs on different spatial scales and much earlier in time.

There are two important implications of these Navier–Stokes results relative to our prior discussion of the asymptotic stages of unsteady separation and possible instabilities. First, because the large-scale interaction in the moderate-Reynolds-number regime occurs before formation of the spike, it is likely that the terminal boundary-layer solution is never reached

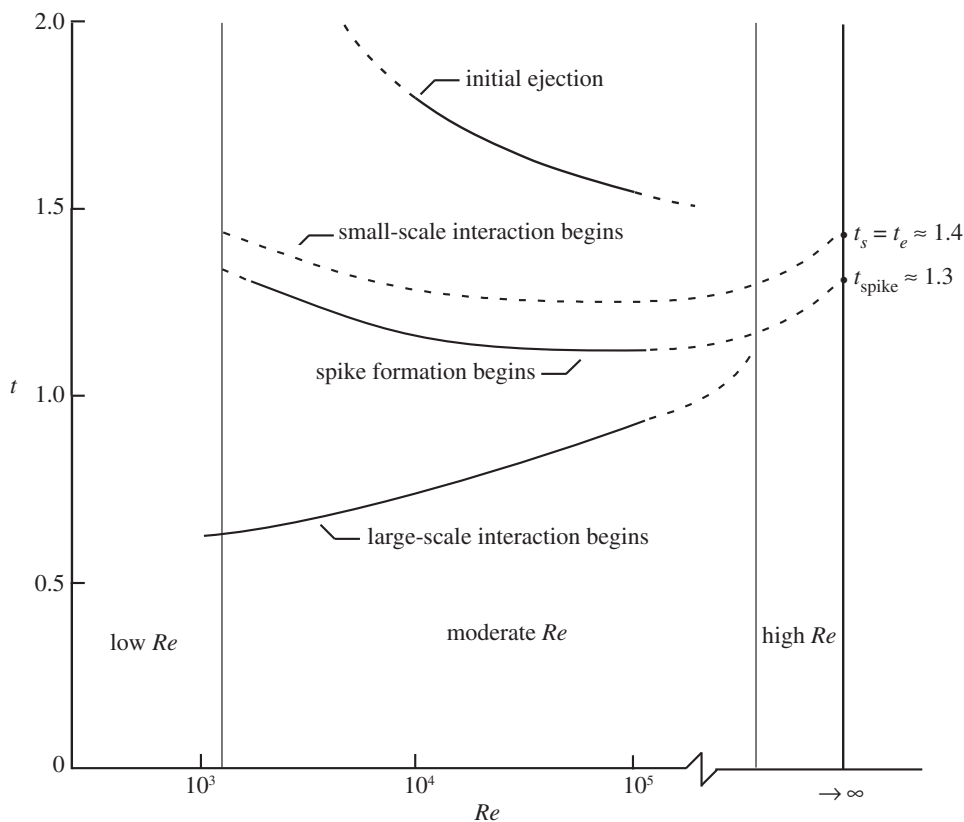


Figure 6. The times at which significant events occur within the three Reynolds-number regimes for the thick-core vortex (from [47]).

in this regime. Second, the fact that interaction begins earlier than predicted by the asymptotic theory in the moderate-Reynolds-number regime suggests that instabilities that may occur in the context of the non-interactive boundary-layer equations, such as the Cowley–Hocking–Tutty instability, may have even less time to manifest themselves prior to the onset of interaction.

By analysing complex singularities in numerical solutions of the boundary-layer and Navier–Stokes equations, a fruitful line of research has been initiated. In particular, although ‘real’ singularities presumably do not occur in the Navier–Stokes solutions, singularities can be tracked in the complex plane. Extending the earlier work of Cowley [52], Rocca *et al.* [53] and Gargano *et al.* [54–56] have tracked complex singularities in non-interactive boundary-layer and Navier–Stokes solutions involving unsteady separation. In the most recent of these investigations, the evolving wall shear stress distribution is analysed for complex singularities. In the context of the boundary-layer solutions, a single complex singularity appears and approaches the real axis as the Van Dommelen and Shen singularity time t_s is approached. In the Navier–Stokes solutions, two additional groups of complex singularities appear corresponding to large- and small-scale interaction. As time evolves, these two groups of singularities move closer to the real axis but never reach it. As the Reynolds number is increased, the streamwise location where these three groups of singularities appear reduces, suggesting that they may collapse into the Van Dommelen and Shen singularity as $Re \rightarrow \infty$. In addition, the temporal gap between the onset of the large- and small-scale interaction diminishes, and the time of formation of large-scale interaction tends towards the time at which viscous–inviscid interaction develops in the boundary-layer equations.

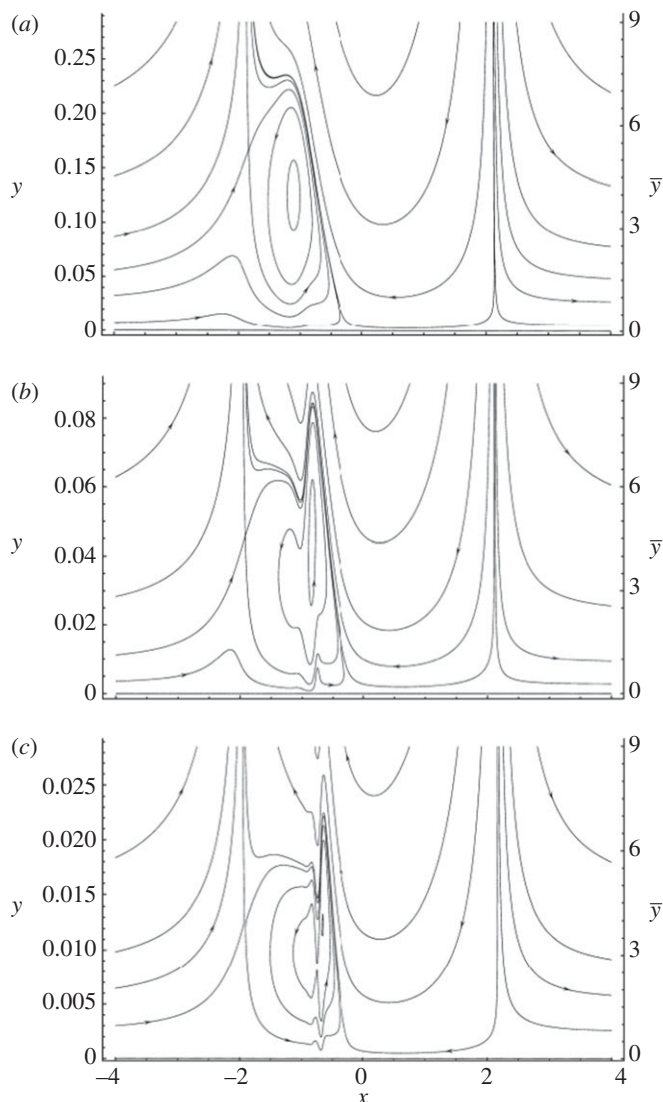


Figure 7. Streamlines within the boundary layer from Navier–Stokes solutions for the thick-core vortex. (a) $Re = 10^3$, $t = 1.4$; (b) $Re = 10^4$, $t = 1.4$; (c) $Re = 10^5$, $t = 1.3$ (from [47]).

(b) Rayleigh instability

Given the high-Reynolds-number shear layers involved, and the instability found in the first interactive stage, it is not surprising that instabilities would appear in Navier–Stokes solutions of these flows. The first sign of such instabilities in two-dimensional unsteady separating flows was provided by Brinckman & Walker [57]. They considered the flow owing to a prescribed spanwise flow induced by longitudinal vortices extending in the streamwise direction, which is a model of the dynamics within a turbulent boundary layer. A high-frequency instability was observed to occur in the ‘alleyway’ produced by an eruption of near-wall vorticity during the unsteady separation process. An estimate of the wavelengths of the oscillations suggests that the instability may be of the Rayleigh type described earlier. However, their results span less than one order of magnitude in Reynolds number. Obabko & Cassel [58], who refer to this instability as the *ejection-induced instability*, observed the same oscillatory behaviour in the moderate-Reynolds-number

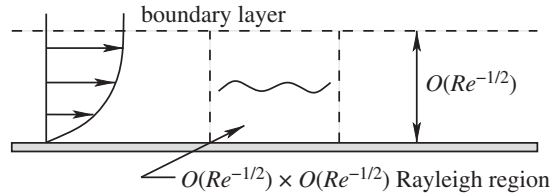


Figure 8. Schematic of the region in which a Rayleigh instability occurs with the boundary layer.

regime using a different Navier–Stokes solver than Brinckman & Walker [57], appearing to lend credence to the presence of the ejection-induced instability. However, the results of Obabko & Cassel [58] show that the oscillations can be eliminated with sufficiently fine spatial resolution. Although the fact that the instability can be eliminated would suggest that it is purely numerical, it may also be the case that a minimum disturbance amplitude, corresponding to the truncation error in the numerical simulations, is required for the instability to be excited within the time frame for which the flow is locally unstable. In any case, another instability was found to occur much earlier in time for higher Reynolds numbers as discussed next.

As highlighted earlier, an inviscid Rayleigh instability is possible in Navier–Stokes solutions, where the wavelengths of the Rayleigh modes are of the same streamwise scale as the thickness of the viscous layer. That is, $1/k \sim O(Re^{-1/2})$ in the boundary-layer case as shown in figure 8. The physics governing these scales occurs within a region having the same scales in the streamwise and normal directions; therefore, the full Navier–Stokes equations must be considered in a stability analysis, and this is why the Rayleigh instability cannot be observed in solutions of the boundary-layer equations. In Navier–Stokes solutions, however, the Rayleigh instability would be expected to dominate over the Cowley–Hocking–Tutty instability as it has a faster growth rate.

The Rayleigh instability within the $O(Re^{-1/2})$ boundary layer occurs on the fast time and short streamwise scales given by $t = Re^{-1/2}T$ and $x = Re^{-1/2}X$, respectively. As shown in Cassel & Obabko [59], a normal mode linear stability analysis shows that the boundary layer becomes unstable to Rayleigh modes when base flow velocity profiles $u_0(\bar{y})$ are such that the Rayleigh equation

$$(u_0 - c) \left[\frac{\partial^2 v_1}{\partial \bar{y}^2} - K^2 v_1 \right] - \frac{\partial^2 u_0}{\partial \bar{y}^2} v_1 = 0 \quad (4.1)$$

produces a complex wavespeed (eigenvalue) having $\text{Im}(c) > 0$. K is the $O(1)$ scaled wavenumber on the Rayleigh scale, such that $k = Re^{1/2}K$, where k is the physical wavenumber, and $\bar{y} = Re^{1/2}y$ is the scaled boundary-layer variable. As is the case in a non-interacting boundary layer, if the streamwise velocity within the boundary layer tends to a constant as $\bar{y} \rightarrow \infty$, Rayleigh’s inflection point theorem is both a necessary and sufficient condition for an instability (in the more general scenario, Rayleigh’s and Fjørtoft’s theorems are necessary, but not sufficient, conditions). The growth rate of the Rayleigh instability is $k\text{Im}(c) = O(Re^{1/2})$ owing to the wavenumber being $k \sim O(Re^{1/2})$. Note that, although the boundary layer is unsteady and may be non-parallel, it appears steady and parallel on the fast time and short streamwise scales of the Rayleigh modes. Thus, the Rayleigh equation arises because of the scalings rather than a parallel flow assumption. It is also the case that, although normal pressure gradients are zero to leading order throughout the boundary layer, the perturbation pressure and its normal derivative play a role within the Rayleigh region.

Given that formation of an inflection point within the streamwise velocity profiles is a necessary and sufficient condition for a Rayleigh instability, one would expect that such instabilities would be common at high Reynolds numbers in regions of adverse pressure gradient. Indeed, solutions of the Rayleigh equation (4.1) for the non-interactive boundary layer induced by a thick-core vortex by Cassel & Obabko [59] show that the flow is unstable to Rayleigh modes

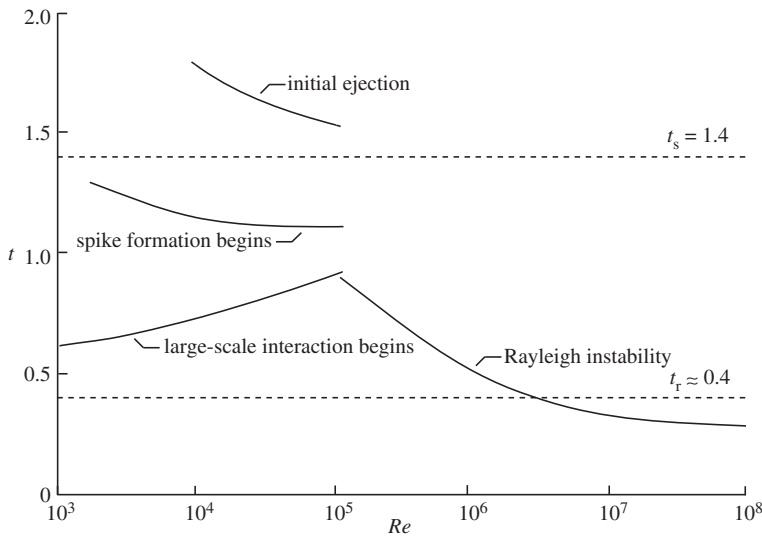


Figure 9. The times at which significant events occur within the Navier–Stokes solutions of the thick-core vortex (from [59]).

for a range of wavenumbers in the region of adverse pressure gradient, where the velocity profiles become inflectional.

Navier–Stokes solutions obtained by Cassel & Obabko [59] for the thick-core vortex confirmed that the Rayleigh instability does indeed exist. Highly resolved and focused spatial grids were required to obtain the Rayleigh instability, which appear in the Navier–Stokes solutions for $Re \geq 10^5$ in the form of oscillations in vorticity and streamwise pressure gradient along the wall in the vicinity of the maximum magnitude of the adverse pressure gradient. Note that changes in the streamwise pressure gradient along the wall require normal pressure gradient effects to become important as predicted by the Rayleigh instability. The dominant wavenumbers (frequencies) were found over a three orders-of-magnitude range of Reynolds numbers with a best-fit analysis revealing a scaling of $k = 0.342 Re^{0.503}$. Note that the exponent closely matches the one-half expected for a Rayleigh instability. In addition, the coefficient in the scaling, i.e. the wavenumber, is found to match remarkably well with that at the streamwise location exhibiting maximum growth rate in the boundary-layer-based Rayleigh-equation calculations. This suggests that the nonlinear and finite-Reynolds-number effects contained within the Navier–Stokes solutions have little effect on the dominant frequencies when compared with the $Re \rightarrow \infty$ predictions. For the highest Reynolds numbers considered, the instability becomes evident in the Navier–Stokes solutions even before formation of the recirculation region and long before interaction begins in the moderate-Reynolds-number regime as illustrated in figure 9, which is an updated version of figure 6. The only distinguishing feature of the flow at such times is the presence of inflectional velocity profiles owing to the adverse streamwise pressure gradient along the surface. Therefore, the Rayleigh instability appears at much earlier times but for higher Reynolds numbers than the injection-induced instability. For the thick-core vortex at least, the high-Reynolds-number regime is never reached in Navier–Stokes solutions unless the Rayleigh instability is suppressed in some manner.

5. Summary: where do we go from here?

Boundary-layer theory has been an active and fruitful area of fluid dynamics research for just over one century, with a dramatic acceleration of activity over the past 40 years. Instrumental in the development of this theory has been the formalization and use of asymptotic methods, in particular singular-perturbation theory. The success of the triple-deck structure in avoiding the

Goldstein singularity in steady separation was surprising, and researchers similarly embarked on a path to resolve the Van Dommelen and Shen singularity that occurs in unsteady separation. Unfortunately, avoiding the Van Dommelen and Shen singularity is not possible using a single asymptotic structure as in the steady case, and the situation is much more complex than was originally thought. Thus, the asymptotic approach has not proved as successful in providing a complete theory of unsteady separation compared with the steady case. The first three asymptotic stages of unsteady separation involve: (i) non-interactive boundary-layer evolution, (ii) first interactive stage, and (iii) normal pressure gradient effects. These successive asymptotic stages appear over ever smaller spatial and shorter temporal scales, and a complete theory that encompasses all of the stages until boundary-layer vorticity is ejected an $O(1)$ distance away from the surface has not yet been forthcoming. Moreover, when the interaction between the boundary layer and the inviscid flow is incorporated, not only is the singularity not removed, but instead occurs sooner than in the non-interaction problem.

Numerical solutions of the Navier–Stokes equations for flows involving unsteady separation have been shown to be broadly consistent with the first three asymptotic stages that arise in the limit as $Re \rightarrow \infty$. However, the second stage, which involves interaction between the viscous boundary layer and inviscid outer flow, exhibits an additional larger-scale interaction earlier in time and over a larger streamwise length scale in finite-Reynolds-number Navier–Stokes solutions when compared with the small-scale interaction that occurs for sufficiently high Reynolds numbers.

In addition to the asymptotic stages that have been identified, unsteady solutions of various asymptotic stages as well as the full Navier–Stokes equations provide for the possibility of various instabilities, which would be expected in such high-Reynolds-number flows. Potential instabilities have been identified during each of the first three asymptotic stages of unsteady separation. In addition, Navier–Stokes solutions involving unsteady separation have confirmed the presence of a Rayleigh instability that may arise as soon as an inflection point appears within a boundary layer in a region of adverse pressure gradient. The fact that the Rayleigh instability arises well before even termination of the first asymptotic stage of unsteady boundary-layer separation calls into question the practical relevance of the asymptotic stages outlined in §2. That is, owing to the presence of the Rayleigh instabilities, is the high-Reynolds-number regime of unsteady separation ever realized in real flows? Alternatively, unlike for numerical solutions of steady problems, where instabilities are inherently suppressed, unsteady solutions of the Navier–Stokes equations involving unsteady separation must contend with the presence of instabilities, such as the Rayleigh instability. Therefore, it raises the question as to how should disturbances be suppressed in order to realize unsteady asymptotic solutions so that limiting solutions as $Re \rightarrow \infty$ give the appropriate unsteady stable solution? Finally, how are these instabilities altered by three-dimensional effects?

When applying asymptotic methods to high-Reynolds-number flows, the hope is that these *laminar* results obtained formally in the limit as $Re \rightarrow \infty$ apply reasonably well down to finite Reynolds numbers below that at which the flow becomes unstable and transitions to turbulence. In the case of unsteady separation, however, the Rayleigh instability pre-empts the evolution predicted by the asymptotic theory for Reynolds number within the moderate-Reynolds-number regime, which is below that required to observe the three stages of unsteady separation in a strict sense. However, the overall features of the non-interactive, interactive and normal pressure gradient stages still are observable within the moderate-Reynolds-number regime when the Rayleigh instability does not occur, notwithstanding the fact that this only spans less than one order of magnitude in Reynolds number below the critical value at which the Rayleigh instability appears for the case of a thick-core vortex.

These results raise the following unanswered questions. What is the relationship between instabilities that arise in the non-interactive and interactive contexts? Recall that non-interacting boundary layers are susceptible to instabilities that are linked to points of zero shear stress, while those that arise in interacting contexts are linked to points of inflection and have faster growth rates. Therefore, does the presence of interaction obviate instabilities associated with points of

zero shear stress? How are the instabilities that arise in asymptotic formulations and the full Navier–Stokes equations related to the actual transition process? What techniques will provide the breakthroughs necessary to articulate a full theoretical description of high-Reynolds-number unsteady separation from inception to breakaway? What is the role of asymptotic methods going forward? What role will global stability (non-parallel effects) and transient growth (non-normal modes) methods play in understanding the instabilities that arise in unsteady separation? What are the three-dimensional effects on the singularities and instabilities found in unsteady separation? How can we prevent these singularities and instabilities from being ‘rediscovered’ as CFD of the Navier–Stokes equations is able to treat increasingly high Reynolds numbers in practical scenarios?

The current state of affairs in our understanding of unsteady separation has a number of important implications for CFD simulations involving high-Reynolds-number boundary-layer flows. It is absolutely necessary that CFD practitioners, who are carrying out increasingly accurate simulations at higher and higher Reynolds numbers, be aware of the important qualitative and quantitative aspects of unsteady separation developed over the years. Although these features occur over very small scales, they have disproportionate effects on overall quantities such as aerodynamic forces. For example, one of the long-standing promises of asymptotics is identification of the temporal and spatial scales that must be resolved in order to accurately simulate realistic flows involving unsteady separation (and other asymptotic behaviour). One thing is clear, whether because of the onset of a succession of singularities occurring on smaller and smaller scales or because of the onset of a Rayleigh instability, the computational demands placed on Navier–Stokes solvers for increasingly high Reynolds numbers are very extreme.

One of the long-term aims of piecing together a complete theory of unsteady separation is the promise of developing efficient and effective means to control such events, once again taking advantage of the disproportionate influence that they have on the overall flow field. Not only do the asymptotics provide knowledge of the dominant physics during the evolution of unsteady separation, the mathematical formulations of the asymptotic stages themselves provide reduced-physics models that can be used in control algorithms. Although this line of research into control of unsteady separation has been pursued experimentally, very little work has been done analytically and/or numerically (see [60,61] and the references therein).

Finally, we provide some reflections on the role that FTS has played in the research that is the subject of this review. FTS was instrumental in elucidating each of the first three asymptotic stages of unsteady separation throughout the 1980s and 1990s. This period perfectly illustrates his manner of carrying out research. In many ways, he served as both the conductor of the orchestra and one of the lead parts; the whole orchestra is necessary to produce a beautiful final product, but things progress much more efficiently and coherently with a capable conductor. FTS not only made significant contributions himself, he enlisted and worked closely with others to further the overall cause. Although FTS primarily focused on analytical approaches, namely asymptotics, he valued very highly, and strongly encouraged, complementary computational studies. For these reasons, FTS’s contributions to this and other fields will no doubt be considered essential and seminal for many years to come. Given the remaining challenges in unsteady separation identified herein, however, we need another FTS to step up and resolve these very difficult theoretical issues and advance our understanding of this most important and ubiquitous phenomenon.

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