Homework #5 – Solutions

Problem 1. (a) A change of variables $y' \to -y'$ in the second integral implies that

$$\bar{f}_\ell^{D/N}(x,y,z) = \frac{1}{(2\pi\ell^2)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{2\ell^2}} f_{\pm}(x',y',z')\, dx'dy'dz'$$

with

$$f_{\pm}(x',y',z') = \Theta(y')f(x',y',z') \mp \Theta(-y')f(x',-y',z')$$

the anti-symmetric/symmetric extensions of f from $\mathbb{R}^3_+$ to $\mathbb{R}^3$.

Since the Gaussian is the fundamental solution of the heat equation

$$\frac{\partial}{\partial\ell^2}\left(\frac{1}{(2\pi\ell^2)^{3/2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\ell^2}\right)\right) = \frac{1}{2}\Delta_x\left(\frac{1}{(2\pi\ell^2)^{3/2}}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\ell^2}\right)\right)$$

and thus

$$\frac{\partial}{\partial\ell^2}\bar{f}_\ell^{D/N} = \frac{1}{2}\Delta_x \bar{f}_\ell^{D/N}.$$

As to boundary conditions,

$$\left[e^{-\frac{(y-y')^2}{2\ell^2}} - e^{-\frac{(y+y')^2}{2\ell^2}}\right]_{y=0} = e^{-y'^2/2\ell^2} - e^{-y'^2/2\ell^2} = 0$$

and thus from the original expression

$$\bar{f}_\ell^{D}(x,0,z) = 0.$$

Similarly,

$$\frac{\partial}{\partial y}\left[e^{-(y-y')^2/2\ell^2} + e^{-(y+y')^2/2\ell^2} \right]$$

$$= \frac{-y}{2\ell}\left(e^{-(y-y')^2/2\ell^2} + e^{-(y+y')^2/2\ell^2} \right)$$

$$+ \frac{y'}{2\ell}\left(e^{-(y-y')^2/2\ell^2} - e^{-(y+y')^2/2\ell^2} \right)$$

so that

$$\frac{\partial}{\partial y}\left[e^{-(y-y')^2/2\ell^2} + e^{-(y+y')^2/2\ell^2} \right]\Bigg|_{y=0}$$

$$= \frac{0}{\ell} \cdot e^{-y'^2/2\ell^2} + \frac{y'}{\ell} \cdot 0 = 0$$

and thus

$$\partial_y \overline{f}_\ell^{N}(x,0,z) = 0 .$$

(b) By integration by parts,

$$\overline{(\partial_y f)}_\ell^{D/N}(x,y,z) = \frac{1}{(2\pi\ell^2)^{3/2}} \int_{\mathbb{R}^3_+} \left[e^{-\frac{(y-y')^2}{2\ell^2}} \mp e^{-\frac{(y+y')^2}{2\ell^2}} \right] e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} \partial_{y'} f(x',y',z')\, dx'dy'dz'$$

(cont'd)

$$= \frac{-1}{(2\pi\ell^2)^{3/2}} \int_{\mathbb{R}^3_+} \partial_{y'} \left[e^{-\frac{(y-y')^2}{2\ell^2}} \mp e^{-\frac{(y+y')^2}{2\ell^2}} \right] e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x',y',z') \, dx'\,dy'\,dz'$$

since $e^{-\frac{(y-y')^2}{2\ell^2}} \mp e^{-\frac{(y+y')^2}{2\ell^2}} = 0$ for $y' = \infty$ and also for $y' = 0$ in the case D ($-$ sign), and $f = 0$ for $y' = 0$ in the case N. However,

$$-\partial_{y'} \left[e^{-\frac{(y-y')^2}{2\ell^2}} \mp e^{-\frac{(y+y')^2}{2\ell^2}} \right] = \partial_y \left[e^{-\frac{(y-y')^2}{2\ell^2}} \pm e^{-\frac{(y+y')^2}{2\ell^2}} \right]$$

so that

$$\overline{(\partial_y f)_\ell}^{D/N}(x,y,z) = \frac{1}{(2\pi\ell^2)^{3/2}} \partial_y \int_{\mathbb{R}^3_+} \left[e^{-\frac{(y-y')^2}{2\ell^2}} \pm e^{-\frac{(y+y')^2}{2\ell^2}} \right] e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x',y',z') \, dx'\,dy'\,dz'$$

$$= \partial_y \overline{f}_\ell^{N/D}(x,y,z),$$

which yields (i), (ii). In that case,

$$[\partial_y f]_\ell^{D/N} := \overline{(\partial_y f)_\ell}^{D/N} - \partial_y \overline{f}_\ell^{D/N}$$

$$= \partial_y \overline{f}_\ell^{N/D} - \partial_y \overline{f}_\ell^{D/N} = \pm \partial_y \overline{f}_\ell^{W}$$

which is the result (iii). Finally, we note that

$$e^{-\frac{(y+y')^2}{2\ell^2}} = e^{-\frac{y^2+2yy'+y'^2}{2\ell^2}} \leq e^{-\frac{y^2+y'^2}{2\ell^2}} \quad \text{for } y, y' \geq 0$$

and thus

$$\left| \bar{f}_\ell^W(x,y,z) \right| \leq \frac{2}{(2\pi\ell^2)^{3/2}} e^{-y^2/2\ell^2}$$
$$\times \int_{\mathbb{R}^3_+} e^{-\frac{y'^2+(x-x')^2+(z-z')^2}{2\ell^2}} |f(x',y',z')| \, dx' dy' dz'.$$

Since we have assumed that f is locally integrable, we see that the latter integral is bounded and thus

$$\bar{f}_\ell^W(x,y,z) = O\left(e^{-y^2/2\ell^2}\right) \ll 1$$

for $y \gg \ell$.

(c) From part (b), (iii)

$$\overline{\left(\partial_y(v\mathbf{u}) + \partial_y p \hat{\mathbf{y}}\right)}_\ell^D = \partial_y \overline{(v\mathbf{u})}_\ell^D + \partial_y \bar{p}_\ell^D \hat{\mathbf{y}}$$
$$+ \left[\partial_y(v\mathbf{u}) + \partial_y p \hat{\mathbf{y}}\right]_\ell^D$$

$$= \partial_y \left[\overline{(v\mathbf{u})}_\ell^D + \bar{p}_\ell^D \hat{\mathbf{y}} \right]$$
$$+ \partial_y \left[\overline{(v\mathbf{u})}_\ell^W + \bar{p}_\ell^W \hat{\mathbf{y}} \right]$$

On the other hand, x- and z-derivatives commute with the heat kernel filter for this geometry, so that

$$\overline{(\partial_x(u\mathbf{u}) + \partial_x p\hat{\mathbf{x}})_\ell}^D = \partial_x\left[\overline{(u\mathbf{u})_\ell}^D + \overline{p}_\ell^D\hat{\mathbf{x}}\right]$$

$$\overline{(\partial_z(w\mathbf{u}) + \partial_z p\hat{\mathbf{z}})_\ell}^D = \partial_z\left[\overline{(w\mathbf{u})_\ell}^D + \overline{p}_\ell^D\hat{\mathbf{z}}\right].$$

Adding these results gives

$$\overline{(\nabla\cdot(\mathbf{u}\mathbf{u} + p\mathbf{I}))_\ell}^D = \nabla\cdot\left(\overline{(\mathbf{u}\mathbf{u})_\ell}^D + \overline{p}_\ell^D\mathbf{I}\right)$$

$$+ \partial_y\left[\overline{(v\mathbf{u})_\ell}^W + \overline{p}_\ell^W\hat{\mathbf{y}}\right].$$

In addition since $\mathbf{u}=\mathbf{0}$ on the boundary $\partial\Omega$ where $y=0$

$$\overline{(\partial_y^2\mathbf{u})_\ell}^D = \partial_y\overline{(\partial_y\mathbf{u})_\ell}^N = \partial_y^2\overline{\mathbf{u}}_\ell^D$$

and thus, together with $\overline{(\partial_x^2\mathbf{u})_\ell}^D = \partial_x^2\overline{\mathbf{u}}_\ell^D$ and the similar result for ∂_z,

$$\overline{(\Delta\mathbf{u})_\ell}^D = \Delta\overline{\mathbf{u}}_\ell^D.$$

Hence, applying the heat kernel filter to the Navier-Stokes equation

$$\partial_t\mathbf{u} + \nabla\cdot(\mathbf{u}\mathbf{u} + p\mathbf{I}) = \nu\Delta\mathbf{u}$$

gives

$$\partial_t \overline{\mathbf{u}}_\ell^D + \nabla \cdot \left[\overline{(\mathbf{u}\mathbf{u})}_\ell^D + \overline{p}_\ell^D \mathbf{I} \right] = \nu \Delta \overline{\mathbf{u}}_\ell^D - \mathbf{f}_\ell^D$$

with

$$\mathbf{f}_\ell^D = \partial_y \left[\overline{(v\mathbf{u})}_\ell^W + \overline{p}_\ell^W \mathbf{I} \right].$$

By exactly analogous arguments,

$$\begin{aligned} \overline{(\partial_y v)}_\ell^D &= \partial_y \overline{v}_\ell^D + [\partial_y v]_\ell^D \\ &= \partial_y \overline{v}_\ell^D + \partial_y \overline{v}_\ell^W \end{aligned}$$

whereas

$$\overline{(\partial_x u)}_\ell^D = \partial_x \overline{u}_\ell^D, \quad \overline{(\partial_z w)}_\ell^D = \partial_z \overline{w}_\ell^D$$

so that adding these terms gives

$$0 = \overline{(\nabla \cdot \mathbf{u})}_\ell^D = \nabla \cdot \overline{\mathbf{u}}_\ell^D + \partial_y \overline{v}_\ell^W$$

$$\Longrightarrow \quad \nabla \cdot \overline{\mathbf{u}}_\ell^D = \sigma_\ell := -\partial_y \overline{v}_\ell^W.$$

Finally, by applying the divergence theorem

$$\int_{\mathbb{R}^3_+} \sigma_\ell \, dV = \int_{\mathbb{R}^3_+} \nabla \cdot \overline{\mathbf{u}}_\ell^D = -\int_{y=0} \overline{v}_\ell^D \, dA = 0$$

since $\overline{v}_\ell^D(x, 0, z) = 0$.

Problem 2. (a) By using spherical coordinates

$$G(\rho) = \frac{1}{(2\pi)^3}\int_0^\infty \kappa^2 d\kappa \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \, \frac{e^{-i\kappa\rho\cos\theta}}{1+\kappa^2}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty \kappa^2 d\kappa \int_{-1}^{1} d\xi \, \frac{e^{-i\kappa\rho\xi}}{1+\kappa^2}, \quad \xi = \cos\theta$$

$$= \frac{1}{(2\pi)^2 \rho}\int_0^\infty \frac{2\kappa \sin(\kappa\rho)}{1+\kappa^2} d\kappa$$

$$= \frac{1}{(2\pi)^2 \rho}\int_{-\infty}^{+\infty} \frac{\kappa \sin(\kappa\rho)}{1+\kappa^2} d\kappa$$

since the integrand is even in κ. In that case,

$$\int_{-\infty}^{+\infty} d\kappa \, \frac{\kappa\sin(\kappa\rho)}{1+\kappa^2} = \operatorname{Im}\left[2\pi i \cdot \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\kappa e^{i\kappa\rho}}{1+\kappa^2}\right]$$

and the integral can be closed in the upper complex half-plane and evaluated by the Cauchy residue theorem from the pole at $\kappa = +i$:

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\kappa e^{i\kappa\rho}}{1+\kappa^2} d\kappa = \frac{1}{2\pi i}\oint_C \frac{\kappa e^{i\kappa\rho}}{(\kappa+i)(\kappa-i)} d\kappa$$

$$= \frac{i e^{-\rho}}{2i} = \frac{1}{2} e^{-\rho}$$

Finally,

$$\int_{-\infty}^{+\infty} d\kappa \, \frac{\kappa \sin(\kappa \rho)}{1+\kappa^2} = \operatorname{Im}\left(\pi i \, e^{-\rho}\right) = \pi e^{-\rho}$$

and thus

$$G(\rho) = \frac{1}{(2\pi)^2 \rho} \int_{-\infty}^{+\infty} \frac{\kappa \sin(\kappa \rho)}{1+\kappa^2} \, d\kappa = \frac{1}{4\pi \rho} e^{-\rho} .$$

(b) From the divergence theorem

$$\int_{\Omega} \frac{\partial}{\partial x_k}\left(\ell^2(x) \frac{\partial \bar{f}_\ell}{\partial x_k} \right) dV = \int_{\partial \Omega} \ell^2(x) \frac{\partial \bar{f}_\ell}{\partial n} \, dA$$

$$= 0$$

since $\ell^2 = 0$ on $\partial\Omega$. Thus, integrating the elliptic equation

$$\bar{f}_\ell - \frac{\partial}{\partial x_k}\left(\ell^2(x) \frac{\partial \bar{f}_\ell}{\partial x_k} \right) = f$$

over the domain Ω, the space-derivative term vanishes and thus

$$\int_{\Omega} \bar{f}_\ell \, dV = \int_{\Omega} f \, dV .$$

(c) Since $\ell^2 = 0$ on $\partial\Omega$,

$$\nabla \ell^2 = \mathbf{n} \frac{\partial \ell^2}{\partial n} \qquad \text{on } \partial\Omega$$

and thus

$$\frac{\partial}{\partial x_k}\left(\ell^2 \frac{\partial \overline{f_\ell}}{\partial x_k} \right) = \frac{\partial \ell^2}{\partial x_k} \frac{\partial \overline{f_\ell}}{\partial x_k} + \ell^2 \frac{\partial^2 \overline{f_\ell}}{\partial x_k^2}$$

gives

$$\left. \frac{\partial}{\partial x_k}\left(\ell^2 \frac{\partial \overline{f_\ell}}{\partial x_k} \right) \right|_{\partial\Omega} = n_k \frac{\partial \ell^2}{\partial n} \frac{\partial \overline{f_\ell}}{\partial x_k} + 0$$

$$= \frac{\partial \ell^2}{\partial n} \frac{\partial \overline{f_\ell}}{\partial n}$$

so that the elliptic equation restricted to the boundary gives

$$\overline{f_\ell} - \frac{\partial \ell^2}{\partial n} \frac{\partial \overline{f_\ell}}{\partial n} = f \qquad \text{on } \partial\Omega .$$

Since $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$,

$$\overline{\mathbf{u}_\ell} = \frac{\partial \ell^2}{\partial n} \frac{\partial \overline{\mathbf{u}_\ell}}{\partial n} \qquad \text{on } \partial\Omega .$$

Thus,

$$\partial \ell^2 / \partial n = 0 \implies \overline{\mathbf{u}_\ell} = \mathbf{0} \qquad \text{on } \partial\Omega$$

$$\partial \ell^2 / \partial n = \Delta_w > 0 \implies \overline{\mathbf{u}_\ell} = \Delta_w \frac{\partial \overline{\mathbf{u}_\ell}}{\partial n} \qquad \text{on } \partial\Omega .$$

(d) Applying the derivative $\frac{\partial}{\partial x_i}$ to the elliptic equation that defines $\bar{f}_\ell$ gives

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\bar{f}_\ell - \frac{\partial}{\partial x_k}\left(\ell^2(x)\frac{\partial \bar{f}_\ell}{\partial x_k}\right)\right]$$

$$= \frac{\partial \bar{f}_\ell}{\partial x_i} - \frac{\partial}{\partial x_k}\left(\ell^2(x)\frac{\partial}{\partial x_k}\left(\frac{\partial \bar{f}_\ell}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial \ell^2}{\partial x_i}\frac{\partial \bar{f}_\ell}{\partial x_k}\right)$$

while the similar equation that defines $\overline{(\partial f/\partial x_i)}_\ell$ is

$$\frac{\partial f}{\partial x_i} = \overline{\left(\frac{\partial f}{\partial x_i}\right)}_\ell - \frac{\partial}{\partial x_k}\left(\ell^2(x)\frac{\partial}{\partial x_k}\overline{\left(\frac{\partial f}{\partial x_i}\right)}_\ell\right).$$

Subtracting the first equation from the second and using the definition

$$\left[\frac{\partial f}{\partial x_i}\right]_\ell = \overline{\left(\frac{\partial f}{\partial x_i}\right)}_\ell - \frac{\partial \bar{f}_\ell}{\partial x_i}$$

gives

$$\left[\frac{\partial f}{\partial x_i}\right]_\ell - \frac{\partial}{\partial x_k}\left(\ell^2(x)\frac{\partial}{\partial x_k}\left[\frac{\partial f}{\partial x_i}\right]_\ell\right) = -\frac{\partial}{\partial x_k}\left(\frac{\partial \ell^2}{\partial x_i}\frac{\partial \bar{f}_\ell}{\partial x_k}\right).$$

By definition of the elliptic filter, this equation implies that

$$\left[\frac{\partial f}{\partial x_i}\right]_\ell = -\overline{\left(\frac{\partial}{\partial x_k}\left(\frac{\partial \ell^2}{\partial x_i}\frac{\partial \bar{f}_\ell}{\partial x_k}\right)\right)}_\ell.$$

(e) Since

$$0 = \overline{(\nabla\cdot u)}_\ell = \nabla\cdot\overline{u}_\ell + [\nabla\cdot u]_\ell,$$

$$\therefore\quad \nabla\cdot\overline{u}_\ell = \sigma_\ell := -[\nabla\cdot u]_\ell = \overline{\left(\frac{\partial}{\partial x_k}\left(\nabla\ell^2\cdot\frac{\partial \overline{u}_\ell}{\partial x_k}\right)\right)}_\ell$$

by part (d). Then,

$$\int_\Omega \sigma_\ell \, dV = \int_\Omega \overline{\left(\frac{\partial}{\partial x_k}\left(\nabla\ell^2\cdot\frac{\partial \overline{u}_\ell}{\partial x_k}\right)\right)}_\ell \, dV$$

$$= \int_\Omega \frac{\partial}{\partial x_k}\left(\nabla\ell^2\cdot\frac{\partial \overline{u}_\ell}{\partial x_k}\right) dV \qquad \text{by part (b)}$$

$$= \int_{\partial\Omega} \nabla\ell^2\cdot\frac{\partial \overline{u}_\ell}{\partial n}\, dA \qquad \text{by the divergence theorem}$$

$$= \int_{\partial\Omega} \Delta_w \mathbf{n}\cdot\frac{\partial \overline{u}_\ell}{\partial n}\, dA \qquad \text{using } \nabla\ell^2 = \Delta_w \mathbf{n} \text{ on } \partial\Omega$$

$$= \int_{\partial\Omega} \Delta_w \frac{\partial u_{\ell,n}}{\partial n}\, dA \qquad \text{w/ } u_{\ell,n} = \overline{u}_\ell\cdot\mathbf{n} \text{ using } \frac{\partial \mathbf{n}}{\partial n} = 0$$

It follows that generally $\int_\Omega \sigma_\ell \, dV \neq 0$ for the elliptic regularization, when $\Delta_w \not\equiv 0$.

Problem 3. (a) We observe that

$$\widetilde{(\nabla f)}_{h,\ell} = \eta_{h,\ell}\overline{(\nabla f)}_\ell = \eta_{h,\ell}\nabla\overline{f}_\ell$$

with $\eta_{h,\ell}(x) = \theta_{h,\ell}(d(x))$, whereas

$$\nabla\widetilde{f}_{h,\ell} = \nabla(\eta_{h,\ell}\overline{f}_\ell) = \eta_{h,\ell}\nabla\overline{f}_\ell + \nabla\eta_{h,\ell}\overline{f}_\ell$$

with $\nabla\eta_{h,\ell} = \theta'_{h,\ell}(d(x))\,\mathbf{n}(x)$ using $\mathbf{n}(x) = \nabla d(x)$. Thus, the derivative commutator for this coarse-graining is

$$[\nabla f]_{h,\ell} := \widetilde{(\nabla f)}_{h,\ell} - \nabla\widetilde{f}_{h,\ell} = -\nabla\eta_{h,\ell}\overline{f}_\ell$$

We can apply this result to derive the equation for $\widetilde{u}_{h,\ell}$ by coarse-graining the Navier-Stokes equation to obtain

$$\partial_t\widetilde{u}_{h,\ell} + \widetilde{(\nabla\cdot[uu + pI - \nu\nabla u])}_{h,\ell} = 0$$

$$\Longrightarrow$$

$$\partial_t\widetilde{u}_{h,\ell} + \nabla\cdot\widetilde{(uu + pI - \nu\nabla u)}_{h,\ell} = -f_{h,\ell}$$

with

$$f_{h,\ell} = [\nabla\cdot(uu + pI - \nu\nabla u)]_{h,\ell}$$

$$= -\nabla\eta_{h,\ell}\cdot\overline{(uu + pI - \nu\nabla u)}_\ell$$

$$= -\theta'_{h,\ell}(d(x))\,\mathbf{n}(x)\cdot\left[\overline{(uu)}_\ell + \overline{p}_\ell I - \nu\nabla\overline{u}_\ell\right]$$

Similarly,

$$0 = \widetilde{(\nabla \cdot \mathbf{u})}_{h,\ell} = \nabla \cdot \tilde{\mathbf{u}}_{h,\ell} + [\nabla \cdot \mathbf{u}]_{h,\ell}$$

$\Longrightarrow$

$$\nabla \cdot \tilde{\mathbf{u}}_{h,\ell} = \sigma_{h,\ell} := -[\nabla \cdot \mathbf{u}]_{h,\ell} = \nabla \eta_{h,\ell} \cdot \overline{\mathbf{u}}_\ell$$

or

$$\nabla \cdot \tilde{\mathbf{u}}_{h,\ell} = \theta'_{h,\ell}(d(\mathbf{x}))\, \mathbf{n}(\mathbf{x}) \cdot \overline{\mathbf{u}}_\ell$$

(b) Applying the divergence theorem,

$$\int_\Omega \sigma_{h,\ell}\, dV = \int_\Omega \nabla \cdot \tilde{\mathbf{u}}_{h,\ell}\, dV$$

$$= \int_{\partial\Omega} \hat{\mathbf{n}} \cdot \tilde{\mathbf{u}}_{h,\ell}\, dA = 0$$

since $\tilde{\mathbf{u}}_{h,\ell} = 0$ within distance h of the boundary. To obtain the Poisson equation for $\tilde{p}_{h,\ell}$, take the divergence of the first equation in part (a) to obtain

$$\partial_t \sigma_{h,\ell} + \nabla\nabla : \left[\widetilde{(\mathbf{u}\mathbf{u})}_{h,\ell} - \nu \widetilde{(\nabla \mathbf{u})}_{h,\ell} \right] + \Delta \tilde{p}_{h,\ell} = -\nabla \cdot \mathbf{f}_{h,\ell}$$

which is rearranged to give

$$-\Delta \tilde{p}_{h,\ell} = \partial_t \sigma_{h,\ell} + \nabla\nabla : \left[\widetilde{(\mathbf{u}\mathbf{u})}_{h,\ell} - \nu \widetilde{(\nabla \mathbf{u})}_{h,\ell} \right] + \nabla \cdot \mathbf{f}_{h,\ell}$$

and note that $\tilde{p}_{h,\ell} = 0$ within distance h of the boundary.

(c) To show that weak solutions satisfying

$$0=\int_0^T dt\int_\Omega dV\left[\partial_t\boldsymbol{\varphi}\cdot\mathbf{u}+\nabla\boldsymbol{\varphi}:\mathbf{u}\mathbf{u}+(\nabla\cdot\boldsymbol{\varphi})p\right]$$

for all $\boldsymbol{\varphi}\in D((0,T)\times\Omega,\mathbb{R}^3)$ are also coarse-grained solutions in the sense of part (a) (for $\nu=0$), we define

$$\boldsymbol{\varphi}_{\mathbf{x},h,\ell,\psi,i}\in D((0,T)\times\Omega,\mathbb{R}^3)$$

$$\boldsymbol{\varphi}_{\mathbf{x},h,\ell,\psi,i}(\mathbf{x}',t')=\eta_{h,\ell}(\mathbf{x})G_\ell(\mathbf{x}'-\mathbf{x})\psi(t')\mathbf{e}_i$$

$$\mathbf{x}\in\Omega,\ 0<\ell<h,\ \psi\in D((0,T)),\ i=1,2,3$$

to obtain

$$0=\int_0^T dt'\left[\partial_{t'}\psi(t')\cdot\tilde{\mathbf{u}}_{h,\ell}(\mathbf{x},t')-\psi(t')\eta_{h,\ell}(\mathbf{x})\nabla\cdot\left(\overline{(\mathbf{u}\mathbf{u})}_\ell(\mathbf{x},t')+\bar{p}_\ell(\mathbf{x},t')\mathbf{I}\right)\right]$$

which is the distributional in time formulation of the equation

$$\partial_t\tilde{\mathbf{u}}_{h,\ell}+\eta_{h,\ell}\nabla\cdot\left(\overline{(\mathbf{u}\mathbf{u})}_\ell+\bar{p}_\ell\mathbf{I}\right)=0$$

or

$$\partial_t\tilde{\mathbf{u}}_{h,\ell}+\nabla\cdot\left(\widetilde{(\mathbf{u}\mathbf{u})}_{h,\ell}+\tilde{p}_{h,\ell}\mathbf{I}\right)$$

$$=\nabla\eta_{h,\ell}\cdot\left(\overline{(\mathbf{u}\mathbf{u})}_\ell+\bar{p}_\ell\mathbf{I}\right)$$

$$=-\mathbf{f}_{h,\ell}$$

To obtain the reverse implication, for any coarse-grained Euler solution $(\mathbf{u},p)$ that satisfies the previous equations for all $0<\ell<h$, we can choose any $\boldsymbol{\varphi}\in D((0,T)\times\Omega,\mathbb{R}^3)$ and smear those equations to obtain by integration by parts

$$0=\int_0^T dt\int_\Omega dV\Big\{\partial_t\boldsymbol{\varphi}(\mathbf{x},t)\cdot\eta_{h,\ell}(\mathbf{x})\overline{\mathbf{u}}_\ell(\mathbf{x},t)$$

$$+\nabla\boldsymbol{\varphi}(\mathbf{x},t):\eta_{h,\ell}(\mathbf{x})\overline{(\mathbf{u}\mathbf{u})}_\ell(\mathbf{x},t)+\nabla\cdot\boldsymbol{\varphi}(\mathbf{x},t)\eta_{h,\ell}(\mathbf{x})\overline{p}_\ell(\mathbf{x},t)$$

$$+\boldsymbol{\varphi}(\mathbf{x},t)\cdot\theta_{h,\ell}'(d(\mathbf{x}))\Big[\overline{(\mathbf{u}\mathbf{u})}_\ell(\mathbf{x},t)+\overline{p}_\ell(\mathbf{x},t)\mathbf{I}\Big]\Big\}.$$

For this fixed $\boldsymbol{\varphi}$ one can take h,ℓ so small that

$$\mathrm{dist}(\mathrm{supp}\,\boldsymbol{\varphi},\partial\Omega)>h+\ell$$

in which case

$$\partial_t\boldsymbol{\varphi}(\mathbf{x},t)\eta_{h,\ell}(\mathbf{x})=\partial_t\boldsymbol{\varphi}(\mathbf{x},t)$$

$$\nabla\boldsymbol{\varphi}(\mathbf{x},t)\eta_{h,\ell}(\mathbf{x})=\nabla\boldsymbol{\varphi}(\mathbf{x},t)$$

$$\boldsymbol{\varphi}(\mathbf{x},t)\theta_{h,\ell}'(d(\mathbf{x}))=\mathbf{0}$$

and therefore, we obtain for these sufficiently small values of $0<\ell<h$ that

$$0 = \int_0^T dt \int_\Omega dV \Big[\partial_t \boldsymbol{\varphi}(\mathbf{x},t) \cdot \overline{\mathbf{u}}_\ell(\mathbf{x},t) + \nabla \boldsymbol{\varphi}(\mathbf{x},t) : \overline{(\mathbf{u}\mathbf{u})}_\ell(\mathbf{x},t) + \nabla \cdot \boldsymbol{\varphi}(\mathbf{x},t)\, \overline{p}_\ell(\mathbf{x},t) \Big]$$

The final step is to take the limit $\ell \to 0$ to obtain

$$0 = \int_0^T dt \int_\Omega dV \Big[\partial_t \boldsymbol{\varphi}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t) + \nabla \boldsymbol{\varphi}(\mathbf{x},t) : \mathbf{u}\mathbf{u}(\mathbf{x},t) + \nabla \cdot \boldsymbol{\varphi}(\mathbf{x},t)\, p(\mathbf{x},t) \Big]$$

for all $\boldsymbol{\varphi} \in D((0,T) \times \Omega, \mathbb{R}^3)$, which is the standard weak formulation of the Euler equations. The weak formulations of the incompressibility constraint

$$\int_0^T dt \int_\Omega \nabla \psi(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t) = 0, \quad \forall \psi \in D((0,T) \times \Omega)$$

$$\Longleftrightarrow \quad \nabla \cdot \tilde{\mathbf{u}}_{h,\ell}(\mathbf{x},t) = \theta'_{h,\ell}(d(\mathbf{x}))\, \mathbf{n}(\mathbf{x}) \cdot \overline{\mathbf{u}}_\ell(\mathbf{x},t)$$

for all $\mathbf{x} \in \Omega$, $0 < \ell < h$
distributionally in time.

The careful discussion of the $\ell \to 0$ can be done assuming that $\mathbf{u} \in L^2((0,T), L^2_{loc}(\Omega))$, $p \in L^1((0,T), L^1_{loc}(\Omega))$ since $\boldsymbol{\varphi}$ has compact support in spacetime and thus one needs only $\lim_{\ell \to 0} \overline{\mathbf{u}}_\ell = \mathbf{u}$, $\lim_{\ell \to 0} \overline{p}_\ell = p$ on this compact set.

Problem 4. (a) Smearing the Navier-Stokes equation with test function $\boldsymbol{\varphi} \in D((0,T)\times\bar{\Omega}, \mathbb{R}^3)$

$$\int_0^T dt \int_\Omega dV\, \boldsymbol{\varphi}(\mathbf{x},t)\cdot\left[\partial_t \mathbf{u}^\nu + \nabla\cdot(\mathbf{u}^\nu\mathbf{u}^\nu) + \nabla p^\nu - \nu\Delta\mathbf{u}^\nu\right] = 0$$

we then use integration by parts

$$\int_0^T dt \int_\Omega dV\, \boldsymbol{\varphi}(\mathbf{x},t)\cdot\partial_t\mathbf{u}^\nu(\mathbf{x},t) = -\int_0^T dt\int_\Omega dV\, \partial_t\boldsymbol{\varphi}(\mathbf{x},t)\cdot\mathbf{u}^\nu(\mathbf{x},t)$$

$$\int_0^T dt \int_\Omega dV\, \boldsymbol{\varphi}(\mathbf{x},t)\cdot\nabla p^\nu(\mathbf{x},t) = -\int_0^T dt\int_\Omega dV\, \nabla\cdot\boldsymbol{\varphi}(\mathbf{x},t)\, p^\nu(\mathbf{x},t)$$
$$-\int_0^T dt\int_{\partial\Omega} dA\, \mathbf{n}\cdot\boldsymbol{\varphi}(\mathbf{x},t)\, p^\nu(\mathbf{x},t),$$

$$\int_0^T dt \int_\Omega dV\, \boldsymbol{\varphi}(\mathbf{x},t)\cdot\left[\nabla\cdot(\mathbf{u}^\nu\mathbf{u}^\nu)(\mathbf{x},t)\right]$$
$$= -\int_0^T dt\int_\Omega dV\, \nabla\boldsymbol{\varphi}(\mathbf{x},t) : \mathbf{u}^\nu(\mathbf{x},t)\,\mathbf{u}^\nu(\mathbf{x},t)$$
$$-\int_0^T dt\int_{\partial\Omega} dA\, \cancelto{0}{(\mathbf{n}\cdot\mathbf{u}^\nu(\mathbf{x},t))}\,(\boldsymbol{\varphi}(\mathbf{x},t)\cdot\mathbf{u}^\nu(\mathbf{x},t))$$

and Green's theorem

$$\int_0^T dt \int_\Omega dV\; \boldsymbol{\varphi}(x,t)\cdot \Delta \mathbf{u}^\nu(x,t) = \int_0^T dt \int_\Omega dV\; \Delta\boldsymbol{\varphi}(x,t)\cdot \mathbf{u}^\nu(x,t)$$

$$= -\int_0^T dt \int_{\partial\Omega} dA \left(\boldsymbol{\varphi}\cdot \frac{\partial \mathbf{u}^\nu}{\partial n} - \cancelto{0}{\mathbf{u}^\nu \cdot \frac{\partial \boldsymbol{\varphi}}{\partial n}} \right)$$

Putting together all of these relations gives

$$\int_0^T dt \int_\Omega dV \left[(\partial_t - \nu\Delta)\boldsymbol{\varphi}\cdot \mathbf{u}^\nu + \nabla\boldsymbol{\varphi} : \mathbf{u}^\nu\mathbf{u}^\nu + (\nabla\cdot\boldsymbol{\varphi})\, p^\nu \right]$$

$$= \int_0^T dt \int_{\partial\Omega} dA \left[-p^\nu (\mathbf{n}\cdot\boldsymbol{\varphi}) + \boldsymbol{\tau}_w^\nu \cdot \boldsymbol{\varphi} \right]$$

using $\boldsymbol{\tau}_w^\nu = \nu \dfrac{\partial \mathbf{u}^\nu}{\partial n}$. This is the desired final result.

Next we note that

$$\left| \int_0^T dt \int_\Omega dV\; \partial_t\boldsymbol{\varphi}\cdot(\mathbf{u}^\nu - \mathbf{u}) \right|$$

$$\leqslant \|\partial_t\boldsymbol{\varphi}\|_{L^2} \, \|\mathbf{u}^\nu - \mathbf{u}\|_{L^2(\mathrm{supp}\,\boldsymbol{\varphi})}$$

and thus

$$\lim_{\nu\to 0} \int_0^T dt \int_\Omega dV\; \partial_t\boldsymbol{\varphi}\cdot\mathbf{u}^\nu = \int_0^T dt \int_\Omega dV\; \partial_t\boldsymbol{\varphi}\cdot\mathbf{u}.$$

An identical argument implies that

$$\lim_{\nu\to 0} \int_0^T dt \int_\Omega dV \, (\nabla\cdot\varphi)\, p^\nu = \int_0^T dt \int_\Omega dV \, (\nabla\cdot\varphi)\, p$$

and

$$\lim_{\nu\to 0} \int_0^T dt \int_\Omega dV \, \Delta\varphi\cdot u^\nu = \int_0^T dt \int_\Omega dV \, \Delta\varphi\cdot u$$

so that

$$\lim_{\nu\to 0} \int_0^T dt \int_\Omega dV \, \nu\Delta\varphi\cdot u^\nu = 0$$

Finally,

$$\int_0^T dt \int_\Omega dV \, \nabla\varphi : (u^\nu u^\nu - uu)$$

$$= \int_0^T dt \int_\Omega dV \, \nabla\varphi : \left[(u^\nu - u) u^\nu + u(u^\nu - u) \right]$$

so that again by Cauchy-Schwartz

$$\left| \int_0^T dt \int_\Omega dV \, \nabla\varphi : (u^\nu u^\nu - uu) \right|$$

$$\leq \|\nabla\varphi\|_{L^\infty} \left[\|u^\nu\|_{L^2(\mathrm{supp}\,\varphi)} + \|u\|_{L^2(\mathrm{supp}\,\varphi)} \right]$$

$$\times \|u^\nu - u\|_{L^2(\mathrm{supp}\,\varphi)} \xrightarrow[\nu\to 0]{} 0$$

Putting together all of these results, we obtain

$$\lim_{\nu \to 0} \int_0^T dt \int_\Omega dV \left[(\partial_t - \nu\Delta)\boldsymbol{\varphi}\cdot\mathbf{u}^\nu + \nabla\boldsymbol{\varphi} : \mathbf{u}^\nu\mathbf{u}^\nu + (\nabla\cdot\boldsymbol{\varphi})p^\nu \right]$$

$$= \int_0^T dt \int_\Omega dV \left[\partial_t\boldsymbol{\varphi}\cdot\mathbf{u} + \nabla\boldsymbol{\varphi} : \mathbf{u}\mathbf{u} + (\nabla\cdot\boldsymbol{\varphi})p \right],$$

as required.

(b) Using the first equation in Problem 3(a) for $\nu = 0$,

$$-\int_0^T dt \int_\Omega dV \, \nabla\eta_{h,\ell}\cdot(\overline{\mathbf{T}}_\ell + \overline{p}_\ell\mathbf{I})\cdot\boldsymbol{\varphi} = \int_0^T dt \int_\Omega dV \, \mathbf{f}_{h,\ell}\cdot\boldsymbol{\varphi}$$

$$= -\int_0^T dt \int_\Omega dV \left[\partial_t\widetilde{\mathbf{u}}_{h,\ell} + \nabla\cdot\left(\widetilde{(\mathbf{u}\mathbf{u})}_{h,\ell} + \widetilde{p}_{h,\ell}\mathbf{I}\right)\right]\cdot\boldsymbol{\varphi}$$

$$= \int_0^T dt \int_\Omega dV \left[\widetilde{\mathbf{u}}_{h,\ell}\cdot\partial_t\boldsymbol{\varphi} + \widetilde{(\mathbf{u}\mathbf{u})}_{h,\ell} : \nabla\boldsymbol{\varphi} + \widetilde{p}_{h,\ell}(\nabla\cdot\boldsymbol{\varphi}) \right]$$

by integration by parts

$$= \int_0^T dt \int_\Omega dV \, \eta_{h,\ell} \left[\overline{\mathbf{u}}_\ell\cdot\partial_t\boldsymbol{\varphi} + \overline{(\mathbf{u}\mathbf{u})}_\ell : \nabla\boldsymbol{\varphi} + \overline{p}_\ell(\nabla\cdot\boldsymbol{\varphi}) \right]$$

by using the definition of $\widetilde{f}_{h,\ell} := \eta_{h,\ell}\overline{f}_\ell$.

Finally, we use the same arguments as in Problem 3(c).

For fixed φ one can take h, ℓ so small that

$$\mathrm{dist}(\mathrm{supp}\,\varphi, \partial\Omega) > h + \ell$$

so that

$$\partial_t \varphi(x,t)\, \eta_{h,\ell}(x) = \partial_t \varphi(x,t)$$

$$\nabla\varphi(x,t)\, \eta_{h,\ell}(x) = \nabla\varphi(x,t).$$

In that case,

$$\lim_{h,\ell\to 0} \int_0^T dt \int_\Omega dV\, \eta_{h,\ell} \left[\overline{u}_\ell \cdot \partial_t \varphi + \overline{(uu)}_\ell : \nabla\varphi + \overline{p}_\ell (\nabla\cdot\varphi) \right]$$

$$= \lim_{\ell\to 0} \int_0^T dt \int_\Omega dV \left[\overline{u}_\ell \cdot \partial_t \varphi + \overline{(uu)}_\ell : \nabla\varphi + \overline{p}_\ell (\nabla\cdot\varphi) \right]$$

$$= \int_0^T dt \int_\Omega dV \left[u \cdot \partial_t \varphi + uu : \nabla\varphi + p(\nabla\cdot\varphi) \right]$$

using the assumptions that $u \in L^2((0,T), L^2_{loc}(\Omega))$, $p \in L^1((0,T), L^1_{loc}(\Omega))$ and the same arguments as in Problem 3(c).

Problem 5. (a) Since $\mathbf{n}(\mathbf{x}) = \nabla d(\mathbf{x})$, we see that

$$\nabla \mathbf{n}(\mathbf{x}) = \nabla \nabla d(\mathbf{x})$$

is a symmetric matrix. Next, note that since $\mathbf{u}\cdot\mathbf{n} = 0$

$$(\mathbf{u}\cdot\nabla)\mathbf{n} \times \mathbf{u} = (\mathbf{u}\cdot\nabla_s)\mathbf{n}\times\mathbf{u}$$

and then the definition of cross product gives

$$\begin{aligned}
\left[(\mathbf{u}\cdot\nabla_s)\mathbf{n}\times\mathbf{u}\right]_i &= \epsilon_{ijk}(u_\ell \partial^s_\ell) n_j u_k \\
&= \epsilon_{ijk} u_\ell (\partial^s_\ell n_j) u_k \\
&= \epsilon_{ijk} u_\ell (\partial^s_j n_\ell) u_k \qquad \text{by symmetry of } \nabla\mathbf{n} \\
&= -\epsilon_{ikj} (u_k \partial^s_j) n_\ell u_\ell \\
&= -\left[(\mathbf{u}\times\nabla_s)\mathbf{n}\cdot\mathbf{u}\right]_i
\end{aligned}$$

or

$$(\mathbf{u}\cdot\nabla_s)\mathbf{n}\times\mathbf{u} = -(\mathbf{u}\times\nabla_s)\mathbf{n}\cdot\mathbf{u} \quad \text{on } \partial\Omega .$$

On the other hand, $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and thus

$$0 = (\mathbf{u}\times\nabla_s)(\mathbf{u}\cdot\mathbf{n}) = (\mathbf{u}\times\nabla_s)\mathbf{u}\cdot\mathbf{n} + (\mathbf{u}\times\nabla_s)\mathbf{n}\cdot\mathbf{u}$$

$\Longrightarrow$

$$(\mathbf{u}\cdot\nabla_s)\mathbf{n}\times\mathbf{u} = (\mathbf{u}\times\nabla_s)\mathbf{u}\cdot\mathbf{n} .$$

(b) Since $\gamma := n \times u$, then $u = \gamma \times n$ and thus

$$u \times \nabla_s = (\gamma \times n) \times \nabla_s .$$

Using the standard vector identity

$$(a \times b) \times c = b\,(a \cdot c) - a\,(b \cdot c)$$

we see that

$$\begin{aligned} u \times \nabla &= (\gamma \times n) \times \nabla \\ &= n\,(\gamma \cdot \nabla) - \gamma\,(n \cdot \nabla) \end{aligned}$$

and for the surface component of ∇, since $n \cdot \nabla_s = 0$,

$$u \times \nabla_s = n\,(\gamma \cdot \nabla_s) = n\,(\gamma \cdot \nabla).$$

Using this relation with the result of part (a)

$$\begin{aligned} (u \cdot \nabla)\, n \times u &= (u \times \nabla_s)\, u \cdot n \\ &= n\big((\gamma \cdot \nabla)\, u \cdot n\big). \end{aligned}$$

Together with the result of the class lecture, we obtain finally that

$$\begin{aligned} -n \times \nabla p &= D_t \gamma - (u \cdot \nabla)\, n \times u \\ &= D_t \gamma - n\big((\gamma \cdot \nabla)\, u \cdot n\big). \end{aligned}$$