Homework No.5, 553.794, Due April 26, 2023.

Problem 1. This problem discusses the *heat-flow regularization* of quantities in the upper half-space $\mathbb{R}^3_+ = \{(x, y, z) : y > 0\}$.

(a) Show for any locally integrable function f defined on $\mathbb{R}^3_+$ that

$$\overline{f}_\ell^{D/N}(x, y, z) := \frac{1}{(2\pi\ell^2)^{3/2}} \int_{\mathbb{R}^3_+} [e^{-\frac{(y-y')^2}{2\ell^2}} \mp e^{-\frac{(y+y')^2}{2\ell^2}}] e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x', y', z')\, dx'dy'dz'$$

solve the heat flow equation $\frac{\partial}{\partial \ell^2}\overline{f}_\ell = \frac{1}{2} \triangle \overline{f}_\ell$ with, respectively, the *Dirichlet b.c*

$$\overline{f}_\ell^D(x, 0, z) = 0.$$

and the *Neumann b.c*

$$\partial_y \overline{f}_\ell^N(x, 0, z) = 0.$$

(b) Use part (a) to show that

$$\begin{array}{ll} (i) & \overline{(\partial_y f)}_\ell^D = \partial_y \overline{f}_\ell^N \\ (ii) & \overline{(\partial_y f)}_\ell^N = \partial_y \overline{f}_\ell^D \quad \text{if } f(x, 0, z) = 0 \\ (iii) & [\partial_y f]_\ell^{D/N} := \overline{(\partial_y f)}_\ell^{D/N} - \partial_y \overline{f}_\ell^{D/N} = \pm \partial_y \overline{f}_\ell^W \end{array}$$

where the third result for case N also requires $f(x, 0, z) = 0$ and we have made the definition $\overline{f}_\ell^W := \overline{f}_\ell^N - \overline{f}_\ell^D$ so that

$$\overline{f}_\ell^W(x, y, z) = \frac{2}{(2\pi\ell^2)^{3/2}} \int_{\mathbb{R}^3_+} e^{-\frac{(y+y')^2}{2\ell^2}} e^{-\frac{(x-x')^2+(z-z')^2}{2\ell^2}} f(x', y', z')\, dx'dy'dz'.$$

Show that $\overline{f}_\ell^W(x, y, z)$ is negligible for $y \gg \ell$.

(c) Use part (b) to derive the coarse-grained Navier-Stokes equations in the half-space

$$\partial_t \overline{\mathbf{u}}_\ell^D + \boldsymbol{\nabla}\boldsymbol{\cdot}[\overline{(\mathbf{uu})}_\ell^D + \overline{p}_\ell^D \mathbf{I}] = \nu \triangle \overline{\mathbf{u}}_\ell^D - \mathbf{f}_\ell^D, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell^D = \sigma_\ell^D$$

with

$$\mathbf{f}_\ell^D := \partial_y [\overline{(v\mathbf{u})}_\ell^W + \overline{p}_\ell^W \hat{\mathbf{y}}], \quad \sigma_\ell^D := -\partial_y \overline{v}_\ell^W.$$

Explain why

$$\int_{\mathbb{R}^3_+} \sigma_\ell^D \, dx\, dy\, dz = 0.$$

Problem 2. This problem discusses the *elliptic regularization* of Germano (1986) and Bose & Moin (2014) defined by the solution $\overline{f}_\ell$ of the equation

$$\overline{f}_\ell - \frac{\partial}{\partial x_k}\left(\ell^2(\mathbf{x})\frac{\partial \overline{f}_\ell}{\partial x_k}\right) = f, \quad \mathbf{x} \in \Omega.$$

(a) When $\Omega = \mathbb{R}^3$ and ℓ^2 is constant, then $\overline{f}_\ell = G_\ell * f$ for $G_\ell(\mathbf{r}) = \ell^{-3}G(\mathbf{r}/\ell)$ and

$$G(\boldsymbol{\rho}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{-i\boldsymbol{\kappa}\boldsymbol{\cdot}\boldsymbol{\rho}}}{1+\kappa^2}\, d^3\kappa.$$

Show that $G(\boldsymbol{\rho}) = e^{-\rho}/4\pi\rho$. *Hint:* Evaluate the above Fourier integral in spherical coordinates using calculus of residues.

(b) If $\ell^2(\mathbf{x})$ is a smooth function vanishing at $\partial\Omega$, then show that

$$\int_\Omega \overline{f}_\ell \, dV = \int_\Omega f\, dV.$$

for any f which is integrable and spatially differentiable.

(c) If $\ell^2(\mathbf{x})$ is a smooth function vanishing at $\partial\Omega$, then show that

$$\overline{f}_\ell - \frac{\partial \ell^2}{\partial n}\frac{\partial \overline{f}_\ell}{\partial n} = f, \quad \mathbf{x} \in \partial\Omega.$$

For a velocity field satisfying $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, show that

$$\overline{\mathbf{u}}_\ell = \mathbf{0} \text{ on } \partial\Omega, \text{ if } \left.\tfrac{\partial \ell^2}{\partial n}\right|_{\partial\Omega} = 0,$$

$$\overline{\mathbf{u}}_\ell = \Delta_w \frac{\partial \overline{\mathbf{u}}_\ell}{\partial n} \text{ on } \partial\Omega, \text{ if } \Delta_w = \left.\tfrac{\partial \ell^2}{\partial n}\right|_{\partial\Omega} > 0.$$

(d) Show that the derivative-commutator for this filter is given in closed form by

$$\left[\frac{\partial f}{\partial x_i}\right]_\ell = -\overline{\left(\frac{\partial}{\partial x_k}\left(\frac{\partial \ell^2}{\partial x_i}\,\frac{\partial \overline{f}_\ell}{\partial x_k}\right)\right)}_\ell$$

(e) Use part (d) to show that $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell = \sigma_\ell$ with

$$\sigma_\ell = \overline{\left(\frac{\partial}{\partial x_k}\left(\boldsymbol{\nabla}\ell^2\boldsymbol{\cdot}\frac{\partial \overline{\mathbf{u}}_\ell}{\partial x_k}\right)\right)}_\ell$$

and therefore when $\boldsymbol{\nabla}\ell^2 = \Delta_w \mathbf{n}$ on $\partial\Omega$

$$\int_\Omega \sigma_\ell \, dV = \int_{\partial\Omega} \Delta_w \frac{\partial \overline{u}_{\ell,n}}{\partial n} dA.$$

Problem 3. This problem discusses the regularization using *filtering & windowing* of Bardos & Titi (2018), defined by

$$\widetilde{f}_{h,\ell}(\mathbf{x}) = \theta_{h,\ell}(d(\mathbf{x}))\overline{f}_\ell(\mathbf{x}), \quad \mathbf{x} \in \Omega, \ h > \ell > 0,$$

for $\theta_{h,\ell}$ the smoothed step-function discussed in the course notes and $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \partial\Omega)$.

(a) Derive the coarse-grained Navier-Stokes equations

$$\partial_t \widetilde{\mathbf{u}}_{h,\ell} + \boldsymbol{\nabla}\boldsymbol{\cdot}[\widetilde{(\mathbf{uu})}_{h,\ell} + \widetilde{p}_{h,\ell}\mathbf{I} - \nu\widetilde{(\boldsymbol{\nabla}\mathbf{u})}_{h,\ell}] = -\mathbf{f}_{h,\ell}, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\widetilde{\mathbf{u}}_{h,\ell} = \sigma_{h,\ell}$$

with

$$\mathbf{f}_{h,\ell} := -\theta'_{h,\ell}(d(\mathbf{x}))\mathbf{n}(\mathbf{x})\boldsymbol{\cdot}[\overline{(\mathbf{uu})}_\ell + \overline{p}_\ell\mathbf{I} - \nu\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell], \quad \sigma_{h,\ell} := \theta'_{h,\ell}(d(\mathbf{x}))\mathbf{n}(\mathbf{x})\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell.$$

(b) Explain why

$$\int_\Omega \sigma_{h,\ell}\, dV = 0$$

and derive the Poisson equation

$$-\triangle\, \widetilde{p}_{h,\ell} = \partial_t\sigma_{h,\ell} + \boldsymbol{\nabla\nabla}\boldsymbol{:}[\widetilde{(\mathbf{uu})}_{h,\ell} - \nu\widetilde{(\boldsymbol{\nabla}\mathbf{u})}_{h,\ell}] + \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{f}_{h,\ell}$$

to determine the coarse-grained pressure with zero Dirichlet boundary conditions.

(c) Explain carefully why the equations in (a) with $\nu = 0$ for all $h > \ell > 0$ are equivalent to the standard weak formulation of the incompressible Euler equations in the flow domain Ω.

Problem 4. (a) If $(\mathbf{u}^\nu, p^\nu)$ is a smooth solution of incompressible Navier-Stokes equations with viscosity ν and if $\boldsymbol{\varphi} \in D((0,T)\times\bar{\Omega}, \mathbb{R}^3)$, then derive the relation

$$\int_0^T dt \int_{\partial\Omega} dA\,[-p^\nu(\mathbf{n}\boldsymbol{\cdot}\boldsymbol{\varphi}) + \boldsymbol{\tau}_w^\nu\boldsymbol{\cdot}\boldsymbol{\varphi}] = \int_0^T dt \int_\Omega dV\,[(\partial_t - \nu\triangle)\boldsymbol{\varphi}\boldsymbol{\cdot}\mathbf{u}^\nu + \boldsymbol{\nabla}\boldsymbol{\varphi}\boldsymbol{:}\mathbf{u}^\nu\mathbf{u}^\nu + (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\varphi})p^\nu].$$

If one assumes strong convergence

$$\mathbf{u}^\nu \to \mathbf{u} \text{ in } L^2((0,T), L^2_{loc}(\Omega)), \quad p^\nu \to p \text{ in } L^1((0,T), L^1_{loc}(\Omega))$$

as $\nu \to 0$, then prove that

$$\int_0^T dt \int_\Omega dV\,[(\partial_t - \nu\triangle)\boldsymbol{\varphi}\boldsymbol{\cdot}\mathbf{u}^\nu + \boldsymbol{\nabla}\boldsymbol{\varphi}\boldsymbol{:}\mathbf{u}^\nu\mathbf{u}^\nu + (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\varphi})p^\nu] \to \int_0^T dt \int_\Omega dV\,[\partial_t\boldsymbol{\varphi}\boldsymbol{\cdot}\mathbf{u} + \boldsymbol{\nabla}\boldsymbol{\varphi}\boldsymbol{:}\mathbf{uu} + (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\varphi})p]$$

(b) If $(\mathbf{u}, p)$ is a solution of the coarse-grained Euler equations in the sense of the equations in Problem 3(a) with $\nu = 0$ for all $h > \ell > 0$ and if $\boldsymbol{\varphi} \in D((0,T) \times \bar{\Omega}, \mathbb{R}^3)$, then derive the relation

$$-\int_0^T dt \int_\Omega dV \, \boldsymbol{\nabla}\eta_{h,\ell} \boldsymbol{\cdot} (\overline{\mathbf{T}}_\ell + \overline{p}_\ell \mathbf{I}) \boldsymbol{\cdot} \boldsymbol{\varphi} = \int_0^T dt \int_\Omega dV \, \eta_{h,\ell}[(\partial_t \boldsymbol{\varphi} \boldsymbol{\cdot} \overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\varphi} \boldsymbol{:} \overline{\mathbf{T}}_\ell + (\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varphi})\overline{p}_\ell].$$

for $\eta_{h,\ell}(\mathbf{x}) := \theta_{h,\ell}(d(\mathbf{x}))$ and $\mathbf{T} := \mathbf{u}\mathbf{u}$. If $\mathbf{u} \in L^2((0,T), L^2_{loc}(\Omega))$, $p \in L^1((0,T), L^1_{loc}(\Omega))$, then prove that

$$\int_0^T dt \int_\Omega dV \, \eta_{h,\ell}[(\partial_t \boldsymbol{\varphi} \boldsymbol{\cdot} \overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\varphi} \boldsymbol{:} \overline{\mathbf{T}}_\ell + (\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varphi})\overline{p}_\ell] \to \int_0^T dt \int_\Omega dV \, [(\partial_t \boldsymbol{\varphi} \boldsymbol{\cdot} \mathbf{u} + \boldsymbol{\nabla}\boldsymbol{\varphi} \boldsymbol{:} \mathbf{u}\mathbf{u} + (\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\varphi})p]$$

as $h, \ell \to 0$.

Problem 5. In this problem we derive the boundary vorticity flux relation for a smooth solution of incompressible Euler equation, in the form

$$-\mathbf{n} \times \boldsymbol{\nabla} p = D_t \boldsymbol{\gamma} - \mathbf{n}((\boldsymbol{\gamma} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{u}) \boldsymbol{\cdot} \mathbf{n}$$

which is invoked in the force field method of Prandtl (1918). Here $\boldsymbol{\gamma} = \mathbf{n} \times \mathbf{u}$ is the strength of the boundary vorticity sheet; we assume that the Euler solution satisfies the no-penetration condition $\mathbf{u} \boldsymbol{\cdot} \mathbf{n} = 0$ so that $\mathbf{u}$ is tangent to the boundary $\partial\Omega$. Our starting point is the alternative vorticity flux relation

$$-\mathbf{n} \times \boldsymbol{\nabla} p = D_t \boldsymbol{\gamma} - (\mathbf{u} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{n} \times \mathbf{u}$$

derived in the course notes.

(a) Show that the Weingarten matrix $\boldsymbol{\nabla}\mathbf{n}$ is symmetric and use this result to derive

$$(\mathbf{u} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{n} \times \mathbf{u} = -(\mathbf{u} \times \boldsymbol{\nabla}_S)\mathbf{n} \boldsymbol{\cdot} \mathbf{u} = (\mathbf{u} \times \boldsymbol{\nabla}_S)\mathbf{u} \boldsymbol{\cdot} \mathbf{n}$$

where $\boldsymbol{\nabla}_S$ is the surface gradient operator, i.e. the component of the gradient tangent to the boundary surface $\partial\Omega$.

(b) Show that

$$\mathbf{u} \times \boldsymbol{\nabla} = \mathbf{n}(\boldsymbol{\gamma} \boldsymbol{\cdot} \boldsymbol{\nabla}) - \boldsymbol{\gamma}(\mathbf{n} \boldsymbol{\cdot} \boldsymbol{\nabla})$$

and use this relation in the result of part (a) to show that

$$(\mathbf{u} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{n} \times \mathbf{u} = \mathbf{n}((\boldsymbol{\gamma} \boldsymbol{\cdot} \boldsymbol{\nabla})\mathbf{u}) \boldsymbol{\cdot} \mathbf{n},$$

completing the derivation.