Homework #4

Problem 1, (a) We calculate that

$$\int_{\mathbb{R}^3} (\nabla\cdot\mathbf{F})\varphi\, dV := -\int_{\mathbb{R}^3} \mathbf{F}\cdot\nabla\varphi\, dV$$

$$= -\int_{\Omega_1} \mathbf{F}_1\cdot\nabla\varphi\, dV - \int_{\Omega_2} \mathbf{F}_2\cdot\nabla\varphi\, dV$$

$$= \int_{\Omega_1} \left[(\nabla\cdot\mathbf{F}_1)\varphi - \nabla\cdot(\mathbf{F}_1\varphi)\right] dV + \int_{\Omega_2} \left[(\nabla\cdot\mathbf{F}_2)\varphi - \nabla\cdot(\mathbf{F}_2\varphi)\right] dV$$

$$= \int_{\mathbb{R}^3} \{\nabla\cdot\mathbf{F}\}\varphi\, dV - \int_{\partial\Omega_1} (\mathbf{n}\cdot\mathbf{F}_1)\varphi\, dA + \int_{\partial\Omega_2} (\mathbf{n}\cdot\mathbf{F}_2)\varphi\, dA$$

by the divergance theorem

$$= \int_{\mathbb{R}^3} \{\nabla\cdot\mathbf{F}\}\varphi\, dV + \int_{\partial\Omega} \mathbf{n}\cdot[\mathbf{F}]\, dA$$

using $[\mathbf{F}] = \mathbf{F}_2 - \mathbf{F}_1$

$$= \int_{\mathbb{R}^3} \left(\{\nabla\cdot\mathbf{F}\} + \mathbf{n}\cdot[\mathbf{F}]\,\delta(d)\right) dV$$

since $\partial\Omega = \{\mathbf{x}\in\mathbb{R}^3 : d(\mathbf{x}) = 0\}$

and $d(d)\, dV = dA$ on $\partial\Omega$

Thus, $\nabla\cdot\mathbf{F} = \{\nabla\cdot\mathbf{F}\} + \mathbf{n}\cdot[\mathbf{F}]\,\delta(d)$.

(b) Likewise, we calculate

$$\int_{\mathbb{R}^3} \nabla\times\mathbf{F}\cdot\boldsymbol{\varphi}\,dV := \int_{\mathbb{R}^3} \mathbf{F}\cdot\nabla\times\boldsymbol{\varphi}\,dV$$

$$= \int_{\Omega_1} \mathbf{F}_1\cdot\nabla\times\boldsymbol{\varphi}\,dV + \int_{\Omega_2} \mathbf{F}_2\cdot\nabla\times\boldsymbol{\varphi}\,dV$$

$$= \int_{\Omega_1} \left[\nabla\times\mathbf{F}_1\cdot\boldsymbol{\varphi} + \nabla\cdot(\boldsymbol{\varphi}\times\mathbf{F}_1)\right] dV$$

$$+ \int_{\Omega_2} \left[\nabla\times\mathbf{F}_2\cdot\boldsymbol{\varphi} + \nabla\cdot(\boldsymbol{\varphi}\times\mathbf{F}_2)\right] dV$$

$$= \int_{\mathbb{R}^3} \{\nabla\times\mathbf{F}\}\cdot\boldsymbol{\varphi}\,dV + \int_{\partial\Omega_1} \mathbf{n}\cdot\boldsymbol{\varphi}\times\mathbf{F}_1\,dA - \int_{\partial\Omega_2} \mathbf{n}\cdot\boldsymbol{\varphi}\times\mathbf{F}_2\,dA$$

by the divergence theorem

$$= \int_{\mathbb{R}^3} \{\nabla\times\mathbf{F}\}\cdot\boldsymbol{\varphi}\,dV + \int_{\partial\Omega} \mathbf{n}\times[\mathbf{F}]\cdot\boldsymbol{\varphi}\,dA$$

using $[\mathbf{F}] = \mathbf{F}_2 - \mathbf{F}_1$

$$= \int_{\mathbb{R}^3} \left(\{\nabla\times\mathbf{F}\} + \mathbf{n}\times[\mathbf{F}]\,\delta(d)\right) dV$$

and thus

$$\nabla\times\mathbf{F} = \{\nabla\times\mathbf{F}\} + \mathbf{n}\times[\mathbf{F}]\,\delta(d).$$

Problem 2. (a) Using the definition of vorticity

$$\int_{\Omega} x_i' \omega_j' \, dV' = \int_{\Omega} x_i' \epsilon_{jkl} \partial_k' u_l' \, dV'$$

$$= -\int_{\Omega} \delta_{ik} \epsilon_{jkl} u_l' \, dV'$$

$$- \int_{\partial B} x_i' \epsilon_{jkl} n_k u_l' \, dA'$$ by integration by parts

$$= \int_{\Omega} \epsilon_{ijl} u_l' \, dV' - \int_{\partial B} x_i' (\mathbf{n} \times \mathbf{u}')_j \, dA'$$

$\Longrightarrow$

$$J_{ij} = \int_{\Omega} x_i' \omega_j' \, dV' + \int_{\partial B} x_i' (\mathbf{n} \times \mathbf{u}')_j \, dA' = \int_{\Omega} \epsilon_{ijl} u_l' \, dV'$$

anti-symmetric in i, j

(b) Using the vector identity $(\mathbf{x}' \times \boldsymbol{\omega}') \times \mathbf{x} = (\mathbf{x}' \cdot \mathbf{x}) \boldsymbol{\omega}' - (\mathbf{x} \cdot \boldsymbol{\omega}') \mathbf{x}'$,

$$\mathbf{I} \times \mathbf{x} = \frac{1}{2} \int_{\Omega} (\mathbf{x}' \times \boldsymbol{\omega}') \times \mathbf{x} \, dV' + \frac{1}{2} \int_{\partial B} (\mathbf{x}' \times (\mathbf{n} \times \mathbf{u}')) \times \mathbf{x} \, dA$$

$$= \frac{1}{2} \mathbf{x}^T \cdot \mathbf{J} - \frac{1}{2} \mathbf{J} \cdot \mathbf{x}$$

$$= \mathbf{x}^T \cdot \mathbf{J} \quad \text{since } \mathbf{J} = -\mathbf{J}^T$$

$$= \int_{\Omega} (\mathbf{x} \cdot \mathbf{x}') \boldsymbol{\omega}' \, dV' + \int_{\partial B} (\mathbf{x} \cdot \mathbf{x}) (\mathbf{n} \times \mathbf{u}') \, dA.$$

Problem 3. (a) Using the definition of vorticity,

$$(x \times \omega)_i = \epsilon_{ijk} x_j \omega_k$$

$$= \epsilon_{ijk} \epsilon_{k\ell m} x_j \partial_\ell u_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) x_j \partial_\ell u_m$$

$$= x_j \partial_i u_j - (x \cdot \nabla) u_i$$

or $$x \times \omega = x_j \nabla u_j - (x \cdot \nabla) u .$$

(b) Using the identity from (a)

$$\int_\Omega x \times \omega \, dV = \int_\Omega (x_j \nabla u_j - (x \cdot \nabla u)) \, dV$$

$$= (-1+3) \int_\Omega u \, dV$$

$$+ \lim_{R \to \infty} \int_{S_R} [(x \cdot u)\hat{x} - (x \cdot \hat{x}) u] \, dA$$

$$- \int_{\partial B} [(x \cdot u) n - (x \cdot n) u] \, dA$$

$$= 2 \int_\Omega u \, dV + \lim_{R \to \infty} \int_{S_R} x \times (\hat{x} \times u) \, dA$$

$$- \int_{\partial B} x \times (n \times u) \, dA \implies$$

$$2\mathbf{I} = \int_{\Omega} \mathbf{x} \times \boldsymbol{\omega} \, dV + \int_{\partial B} \mathbf{x} \times (\mathbf{n} \times \mathbf{u}) \, dA$$

$$= \underbrace{2 \int_{\Omega} \mathbf{u} \, dV}_{2\mathbf{P}} + \lim_{R \to \infty} \int_{S_R} \mathbf{x} \times (\hat{\mathbf{x}} \times \mathbf{u}) \, dA$$

(c) From the leading term in the asymptotic multipole expansion

$$\mathbf{u}(\mathbf{x}) \sim -\frac{\mathbf{I}}{4\pi r^3} + \frac{3(\mathbf{I} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}}{4\pi r^3} + O\left(\frac{1}{r^4}\right)$$

one can see that

$$\hat{\mathbf{x}} \times \mathbf{u} \sim \frac{\mathbf{I} \times \hat{\mathbf{x}}}{4\pi r^3} + O\left(\frac{1}{r^4}\right)$$

and thus

$$\mathbf{x} \times (\hat{\mathbf{x}} \times \mathbf{u}) \sim \frac{\hat{\mathbf{x}} \times (\mathbf{I} \times \hat{\mathbf{x}})}{4\pi r^2} + O\left(\frac{1}{r^3}\right)$$

$$= \frac{\mathbf{I} - (\mathbf{I} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}}{4\pi r^2} + O\left(\frac{1}{r^3}\right)$$

$$\Longrightarrow \lim_{R \to \infty} \int_{S_R} \mathbf{x} \times (\hat{\mathbf{x}} \times \mathbf{u}) \, dA = \lim_{R \to \infty} \int_{S_R} \frac{\mathbf{I} - (\mathbf{I} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}}{4\pi R^2} R^2 \, d\Omega = \frac{2}{3} \mathbf{I}$$

using $\frac{1}{4\pi} \int_{S_R} \hat{x}_i \hat{x}_j \, d\Omega = \frac{1}{3} \delta_{ij}$. From $2\mathbf{I} = 2\mathbf{P} + \frac{2}{3}\mathbf{I}$, one easily gets

$$\mathbf{P} = \frac{2}{3} \mathbf{I}.$$

Problem 4. (a) We observe that

$$\int_{\Omega} \mathbf{u}_\phi \cdot \mathbf{u} \times \boldsymbol{\omega} \, dV = \int_{\Omega} \mathbf{u}_\phi \cdot \mathbf{u}_\omega \times \boldsymbol{\omega} \, dV$$

$$= \int_{\Omega} \mathbf{u}_\phi \cdot \left[-\nabla \cdot (\mathbf{u}_\omega \mathbf{u}_\omega) + \nabla\left(\tfrac{1}{2}|\mathbf{u}_\omega|^2\right) \right] dV$$

$$= \lim_{R \to \infty} \int_{\Omega_R} \mathbf{u}_\phi \cdot \left[-\nabla \cdot (\mathbf{u}_\omega \mathbf{u}_\omega) + \nabla\left(\tfrac{1}{2}|\mathbf{u}_\omega|^2\right) \right] dV$$

$$\Omega_R = \Omega \cap B_R(0)$$

$$= \int_{\Omega} \left[\nabla \mathbf{u}_\phi : \mathbf{u}_\omega \mathbf{u}_\omega - \cancelto{0}{(\nabla \cdot \mathbf{u}_\phi)} \tfrac{1}{2}|\mathbf{u}_\omega|^2 \right] dV$$

$$+ \int_{\partial B} \left[\cancelto{0}{(\mathbf{n} \cdot \mathbf{u}_\omega)} (\mathbf{u}_\omega \cdot \mathbf{u}_\phi) - \cancelto{0}{(\mathbf{n} \cdot \mathbf{u}_\phi)} \tfrac{1}{2}|\mathbf{u}_\omega|^2 \right] dA$$

$$- \lim_{R \to \infty} \int_{S_R} \left[(\hat{\mathbf{x}} \cdot \mathbf{u}_\omega)(\mathbf{u}_\omega \cdot \mathbf{u}_\phi) - (\hat{\mathbf{x}} \cdot \mathbf{u}_\phi) \tfrac{1}{2}|\mathbf{u}_\omega|^2 \right] dA.$$

Note that the last limit vanishes, since

$$\mathbf{u}_\phi \sim \mathbf{V}, \quad \mathbf{u}_\omega = O\left(\frac{1}{r^3}\right), \quad r \to \infty$$

and thus

$$\int_{\Omega} \mathbf{u}_\phi \cdot \mathbf{u} \times \boldsymbol{\omega} \, dV = \int_{\Omega} \nabla \mathbf{u}_\phi : \mathbf{u}_\omega \mathbf{u}_\omega \, dV.$$

(b) Note that

$$\nabla \cdot (\boldsymbol{\omega} \times \mathbf{u}_\phi) = \mathbf{u}_\phi \cdot \nabla \times \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \cancelto{0}{\nabla \times \mathbf{u}_\phi}$$

and thus

$$\int_\Omega \mathbf{u}_\phi \cdot \nu \nabla \times \boldsymbol{\omega} \, dV = \nu \int_\Omega \nabla \cdot (\boldsymbol{\omega} \times \mathbf{u}_\phi) \, dV$$

$$= \lim_{R \to \infty} \nu \int_{\Omega_R} \nabla \cdot (\boldsymbol{\omega} \times \mathbf{u}_\phi) \, dV$$

$$= -\nu \int_{\partial B} \mathbf{n} \cdot (\boldsymbol{\omega} \times \mathbf{u}_\phi) \, dA$$

$$+ \lim_{R \to \infty} \nu \int_{S_R} \hat{\mathbf{x}} \cdot (\boldsymbol{\omega} \times \mathbf{u}_\phi) \, dA.$$

The last limit vanishes again because $\mathbf{u}_\phi \sim \mathbf{V}$, $\boldsymbol{\omega} = O\left(\frac{1}{r^4}\right)$ as $r \to \infty$. We conclude that

$$\int_\Omega \mathbf{u}_\phi \cdot \nu \nabla \times \boldsymbol{\omega} \, dV = \int_{\partial \Omega} \mathbf{u}_\phi \cdot (\nu \boldsymbol{\omega} \times \mathbf{n}) \, dA$$

$$= \int_{\partial \Omega} \mathbf{u}_\phi \cdot \boldsymbol{\tau}_w \, dA$$

since $\boldsymbol{\tau}_w = \nu \boldsymbol{\omega} \times \mathbf{n}$.

Problem 5. (a) Note from the definition

$$u(x,t) = \bar{u}(x + X(t), t) - V(t)$$

and the chain rule that

$$\begin{aligned}
\partial_t u &= \partial_t \bar{u} + (V(t)\cdot\bar{\nabla})\bar{u} - A(t) \\
&= \left(-(\bar{u}\cdot\bar{\nabla})\bar{u} - \bar{\nabla}\bar{p} + \nu\bar{\Delta}\bar{u}\right) \\
&\quad + (V(t)\cdot\bar{\nabla})\bar{u} - A(t) \\
&= -((\bar{u} - V)\cdot\bar{\nabla})\bar{u} - \bar{\nabla}(\bar{p} + A(t)\cdot x) + \nu\bar{\Delta}\bar{u} \\
&= -(u\cdot\nabla)u - \nabla p + \nu\Delta u
\end{aligned}$$

using in the last step the definition

$$p(x,t) = \bar{p}(x + X(t), t) + A(t)\cdot x.$$

Furthermore,

$$u\big|_{\partial B} = \bar{u}\big|_{\partial B(t)} - V(t) = V(t) - V(t) = 0$$

and

$$\lim_{|x|\to\infty} u = \lim_{|x|\to\infty} \bar{u} - V = 0 - V = -V$$

(b) Note from the definition

$$\phi(\mathbf{x},t) = \overline{\Phi}(\mathbf{x}+\mathbf{X}(t),t) - \mathbf{V}(t)\cdot\mathbf{x}$$

and the chain rule that

$$\partial_t \phi = \partial_t \overline{\Phi} + (\mathbf{V}(t)\cdot\overline{\nabla})\overline{\Phi} - \mathbf{A}(t)\cdot\mathbf{x}$$

$$= \left(-\tfrac{1}{2}|\overline{\nabla}\overline{\Phi}|^2 - \overline{P}_\phi + \overline{c}(t)\right)$$

$$+ (\mathbf{V}(t)\cdot\overline{\nabla})\overline{\Phi} - \mathbf{A}(t)\cdot\mathbf{x}.$$

Since $\nabla\phi = \overline{\nabla}\overline{\Phi} - \mathbf{V}(t)$,

$$\tfrac{1}{2}|\nabla\phi|^2 = \tfrac{1}{2}|\overline{\nabla}\overline{\Phi}|^2 - \mathbf{V}(t)\cdot\overline{\nabla}\overline{\Phi} + \tfrac{1}{2}|\mathbf{V}(t)|^2$$

and thus we obtain the Bernoulli equation for ϕ

$$\partial_t\phi = \left(-\tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}|\mathbf{V}(t)|^2\right) - \left(\overline{P}_\phi + \mathbf{A}(t)\cdot\mathbf{x}\right) + \overline{c}(t)$$

$$= -\tfrac{1}{2}|\nabla\phi|^2 - P_\phi + c(t)$$

with $P_\phi(\mathbf{x},t) = \overline{P}_\phi(\mathbf{x}+\mathbf{X}(t),t) + \mathbf{A}(t)\cdot\mathbf{x}$ and $c(t) = \overline{c}(t) + \tfrac{1}{2}|\mathbf{V}(t)|^2$.

Finally,

$$\left.\frac{\partial\phi}{\partial n}\right|_{\partial B} = \left.\frac{\partial\overline{\Phi}}{\partial n}\right|_{\partial B(t)} - \mathbf{V}(t)\cdot\mathbf{n} = \mathbf{V}(t)\cdot\mathbf{n} - \mathbf{V}(t)\cdot\mathbf{n} = 0$$

and

$$\phi = \overline{\Phi} - \mathbf{V}(t)\cdot\mathbf{x} \sim 0 - \mathbf{V}(t)\cdot\mathbf{x} = -\mathbf{V}(t)\cdot\mathbf{x}$$

as $|\mathbf{x}|\to\infty$, and, of course, $\Delta\phi = \overline{\Delta}\overline{\Phi} = 0$.

Problem 6. (a) Note that

$$\int_{\Omega} \mathbf{u}_\phi \, dV = \lim_{R\to\infty} \int_{\Omega_R} \nabla\phi \, dV$$

$$= -\int_{\partial B} \mathbf{n}\phi \, dA + \lim_{R\to\infty} \int_{S_R} \hat{\mathbf{x}}\phi \, dA$$

but the latter limit diverges, because $\phi \sim -\mathbf{V}(t)\cdot\mathbf{x}$ as $|\mathbf{x}| \to \infty$.

Defining $\mathbf{P}_\phi = -\int_{\partial B} \phi \, \mathbf{n} \, dA$, we obtain from the Bernoulli relation

$$\frac{d\mathbf{P}_\phi}{dt} = -\int_{\partial B} \partial_t \phi \, \mathbf{n} \, dA$$

$$= \int_{\partial B} \left(p_\phi + \tfrac{1}{2}|\mathbf{u}_\phi|^2 - \cancel{c(t)}^{\,0}\right)\mathbf{n} \, dA$$

(b) Note first that

$$\nabla \mathbf{u}_\phi = \nabla\nabla\phi$$

is symmetric and thus

$$\partial_i \left(\tfrac{1}{2}|\mathbf{u}_\phi|^2\right) = \mathbf{u}_\phi \cdot \partial_i \mathbf{u}_\phi = \mathbf{u}_\phi \cdot \nabla u_{\phi i}$$

or

$$\nabla\left(\tfrac{1}{2}|\mathbf{u}_\phi|^2\right) = (\mathbf{u}_\phi\cdot\nabla)\mathbf{u}_\phi$$

Thus,

$$\int_{\Omega_R} \nabla\left(\tfrac{1}{2}|\mathbf{u}_\phi|^2\right) dV = \int_{\Omega_R} (\mathbf{u}_\phi\cdot\nabla)\mathbf{u}_\phi \, dV$$

implying that

$$- \int_{\partial B} \mathbf{n} \, \tfrac{1}{2} |\mathbf{u}_\phi|^2 \, dA + \int_{S_R} \hat{\mathbf{x}} \, \tfrac{1}{2} |\mathbf{u}_\phi|^2 \, dA$$

$$= - \int_{\partial B} \overset{0}{\cancel{(\mathbf{u}_\phi \cdot \mathbf{n})}} \mathbf{u}_\phi \, dA + \int_{S_R} (\mathbf{u}_\phi \cdot \hat{\mathbf{x}}) \mathbf{u}_\phi \, dA .$$

From the proof of the d'Alembert theorem, we know that

$$\mathbf{u}_\phi = - \mathbf{V}(t) + \tilde{\mathbf{u}}_\phi , \quad \tilde{\mathbf{u}}_\phi = O\left(\frac{1}{r^3}\right) \text{ as } r \to \infty .$$

Since $\int_{S_R} \hat{\mathbf{x}} \, dA = 0$, we obtain

$$\lim_{R \to \infty} \int_{S_R} \hat{\mathbf{x}} \, \tfrac{1}{2} |\mathbf{u}_\phi|^2 \, dA = \lim_{R \to \infty} - \int_{S_R} \hat{\mathbf{x}} \, (\mathbf{V}(t) \cdot \tilde{\mathbf{u}}_\phi) \, dA = 0$$

$$\lim_{R \to \infty} \int_{S_R} (\mathbf{u}_\phi \cdot \hat{\mathbf{x}}) \mathbf{u}_\phi \, dA = \lim_{R \to \infty} - \int_{S_R} \left[(\mathbf{V}(t) \cdot \hat{\mathbf{x}}) \tilde{\mathbf{u}}_\phi + (\tilde{\mathbf{u}}_\phi \cdot \hat{\mathbf{x}}) \mathbf{V}(t) \right] dA$$

$$= 0$$

and we thus conclude that

$$\int_{\partial B} \tfrac{1}{2} |\mathbf{u}_\phi|^2 \mathbf{n} \, dA = 0 .$$

(c) From parts (a) and (b) we thus obtain that

$$\frac{d\mathbf{P}_\phi(t)}{dt} = \int_{\partial B} P_\phi \, \mathbf{n} \, dA = \mathbf{F}_\phi .$$

Recalling that ϕ satisfies

$$-\Delta\phi = 0 \quad \text{in } \Omega$$

$$\frac{\partial \phi}{\partial n} = -\mathbf{V}(t)\cdot\mathbf{n} \quad \text{on } \partial B$$

$$\phi \longrightarrow 0 \quad \text{as } |\mathbf{x}| \longrightarrow \infty,$$

then it follows from the theory of the Neumann problem for the Laplace equation that ϕ will be bounded whenever $\mathbf{V}(t)$ is bounded. In that case,

$$\begin{aligned}
\overline{\mathbf{F}}_\phi &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{F}_\phi(t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{d\mathbf{P}_\phi}{dt}(t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\left[\mathbf{P}_\phi(T) - \mathbf{P}_\phi(0)\right] = \mathbf{0},
\end{aligned}$$

so that the long-time average of the force vanishes.