Homework No.4, 553.794, Due April 10, 2023.

Problem 1. This problem considers a *piecewise smooth* vector field $\mathbf{F}$ defined for a simply-connected open set Ω_1 with smooth boundary $\partial\Omega = \partial\Omega_1$ and simply-connected open complement $\Omega_2 = \mathbb{R}^3 \backslash \bar{\Omega}_1$ such that

$$\mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}_1(\mathbf{x}) & \mathbf{x} \in \Omega_1 \\ \mathbf{F}_2(\mathbf{x}) & \mathbf{x} \in \Omega_2 \end{cases}$$

with smooth $\mathbf{F}_1 : \Omega_1 \to \mathbb{R}^3$ and $\mathbf{F}_2 : \Omega_2 \to \mathbb{R}^3$.

(a) Defining the distributional divergence of $\mathbf{F}$ by

$$\int (\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F})\varphi \, dV = -\int \mathbf{F}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi \, dV$$

for a C^∞ and rapidly decaying scalar test function φ, show that

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = \{\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}\} + \mathbf{n}\boldsymbol{\cdot}[\mathbf{F}]\delta(d)$$

where

$$\{\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}\}(\mathbf{x}) = \begin{cases} \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}_1(\mathbf{x}) & \mathbf{x} \in \Omega_1 \\ \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}_2(\mathbf{x}) & \mathbf{x} \in \Omega_2, \end{cases}$$

and where $d(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \partial\Omega)$, $[\mathbf{F}] = \mathbf{F}_2 - \mathbf{F}_1$ on $\partial\Omega$, and $\mathbf{n}$ is the unit normal on $\partial\Omega$ pointing from Ω_1 into Ω_2.

(a) Defining similarly the distributional curl of $\mathbf{F}$ by

$$\int (\boldsymbol{\nabla}\times\mathbf{F})\boldsymbol{\cdot}\boldsymbol{\varphi} \, dV = \int \mathbf{F}\boldsymbol{\cdot}(\boldsymbol{\nabla}\times\boldsymbol{\varphi}) \, dV$$

for a C^∞ and rapidly decaying vector test function $\boldsymbol{\varphi}$, show that

$$\boldsymbol{\nabla}\times\mathbf{F} = \{\boldsymbol{\nabla}\times\mathbf{F}\} + \mathbf{n}\times[\mathbf{F}]\delta(d)$$

where

$$\{\boldsymbol{\nabla}\times\mathbf{F}\}(\mathbf{x}) = \begin{cases} \boldsymbol{\nabla}\times\mathbf{F}_1(\mathbf{x}) & \mathbf{x} \in \Omega_1 \\ \boldsymbol{\nabla}\times\mathbf{F}_2(\mathbf{x}) & \mathbf{x} \in \Omega_2, \end{cases}$$

and all other definitions are the same as in part (a).

Problem 2. We study results relevant to the *multipole expansion* for the vector potential $\boldsymbol{\psi}(\mathbf{x})$ of a differentiable, conditionally integrable, solenoidal velocity field $\mathbf{u}(\mathbf{x})$ in the domain $\Omega = \mathbb{R}^3 \backslash B$ outside a smooth, simply-connected body B.

(a) Prove that the integral involving the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$

$$\int_\Omega x_i' \omega_j' \, dV' + \int_{\partial B} x_i' (\mathbf{n}\times\mathbf{u})_j' \, dA'$$

is anti-symmetric in i and j.

(b) Use the result in (a) to show that the *vector impulse* defined by

$$\mathbf{I} = \frac{1}{2} \int_\Omega \mathbf{x}'\times\boldsymbol{\omega}' \, dV' + \frac{1}{2} \int_{\partial B} \mathbf{x}'\times(\mathbf{n}\times\mathbf{u})' \, dA'$$

satisfies

$$\mathbf{I}\times\mathbf{x} = \int_\Omega (\mathbf{x}\cdot\mathbf{x}')\boldsymbol{\omega}' \, dV' + \int_{\partial B} (\mathbf{x}\cdot\mathbf{x}')(\mathbf{n}\times\mathbf{u})' \, dA'.$$

Problem 3. This problem gives a simple derivation of the relation between momentum and impulse, with the same assumptions on velocity $\mathbf{u}(\mathbf{x})$ as Problem 2.

(a) Derive the following identity involving the vorticity:

$$\mathbf{x}\times\boldsymbol{\omega} = x_i \boldsymbol{\nabla} u_i - (\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{u}.$$

(b) Use the result in part (a) to show that impulse $\mathbf{I}$ and momentum $\mathbf{P} = \int_\Omega \mathbf{u} \, dV$ are related by

$$2\mathbf{I} = 2\mathbf{P} + \lim_{R\to\infty} \int_{S_R} \mathbf{x}\times(\hat{\mathbf{x}}\times\mathbf{u}) \, dA,$$

where S_R is the sphere of radius R centered at the origin.

(c) Using the asymptotic far-field expansion

$$\mathbf{u}(\mathbf{x}) \sim \frac{-\mathbf{I}r^2 + 3(\mathbf{I}\cdot\mathbf{x})\mathbf{x}}{4\pi r^5}, \quad r \to \infty,$$

show that

$$\lim_{R\to\infty} \int_{S_R} \mathbf{x}\times(\hat{\mathbf{x}}\times\mathbf{u}) \, dA = \frac{2}{3}\mathbf{I}$$

and conclude from part (b) that $\mathbf{P} = \frac{2}{3}\mathbf{I}$.

Problem 4. With the same notations and assumptions as in the derivation of the Josephson-Anderson relation for external flow around a smooth body B, derive the following alternative expressions:

(a) $$\int_\Omega \mathbf{u}_\phi\boldsymbol{\cdot}\mathbf{u}\times\boldsymbol{\omega}\, dV = \int_\Omega \boldsymbol{\nabla}\mathbf{u}_\phi\boldsymbol{:}\mathbf{u}_\omega\mathbf{u}_\omega\, dV$$

(b) $$\int_\Omega \mathbf{u}_\phi\boldsymbol{\cdot}\nu\boldsymbol{\nabla}\times\boldsymbol{\omega}\, dV = \int_{\partial B} \mathbf{u}_\phi\boldsymbol{\cdot}\boldsymbol{\tau}_w\, dA.$$

Carefully justify the neglect of boundary terms in integration by parts.

Problem 5. We consider in this problem the general translational motion of a solid body through an incompressible fluid at rest at infinity. The body is represented by the time-dependent set

$$B(t) = B + \mathbf{X}(t)$$

where $\mathbf{X} : [0, T] \to \mathbb{R}^3$ is a smooth function with $\mathbf{X}(0) = \mathbf{0}$ and B is a simply-connected open set with a smooth boundary ∂B. Set $\mathbf{V}(t) = \dot{\mathbf{X}}(t)$ and $\mathbf{A}(t) = \ddot{\mathbf{X}}(t)$.

(a) The incompressible Navier-Stokes solution $(\bar{\mathbf{u}}(\bar{\mathbf{x}}, t), \bar{p}(\bar{\mathbf{x}}, t))$ in the space domain $\Omega(t) = \mathbb{R}^3\backslash B(t)$ for the fluid reference frame satisfies the boundary conditions

$$\bar{\mathbf{u}} = \mathbf{V}(t) \quad \text{on } \partial B(t); \quad \bar{\mathbf{u}} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty.$$

Show that the transformations

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x} + \mathbf{X}(t), t) - \mathbf{V}(t), \quad p(\mathbf{x}, t) = \bar{p}(\mathbf{x} + \mathbf{X}(t), t) + \mathbf{A}(t)\boldsymbol{\cdot}\mathbf{x}$$

give the solution of the incompressible Navier-Stokes equation in the space domain $\Omega = \mathbb{R}^3\backslash B$ for the body frame, which satisfies the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial B; \quad \mathbf{u} \to -\mathbf{V}(t) \quad \text{as } |\mathbf{x}| \to \infty.$$

(b) For the same situation as in part (a), consider the potential solution $\bar{\mathbf{u}}_\phi = \bar{\boldsymbol{\nabla}}\bar{\phi}$ of the incompressible Euler equations in the fluid frame, with $\bar{\phi}(\bar{\mathbf{x}}, t)$ solving the Laplace equation $\bar{\triangle}\bar{\phi} = 0$ in $\Omega(t)$ with boundary conditions

$$\frac{\partial\bar{\phi}}{\partial n} = \mathbf{V}(t)\boldsymbol{\cdot}\mathbf{n} \quad \text{on } \partial B(t); \quad \bar{\phi} \to 0 \quad \text{as } |\mathbf{x}| \to \infty,$$

and with pressure $\bar{p}_\phi(\bar{\mathbf{x}}, t)$ given by the Bernoulli equation

$$\partial_t\bar{\phi} + \frac{1}{2}|\bar{\boldsymbol{\nabla}}\bar{\phi}|^2 + \bar{p}_\phi = \bar{c}(t)$$

for some arbitrary function $\bar{c}(t)$. Show that the transformations

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x} + \mathbf{X}(t), t) - \mathbf{V}(t)\boldsymbol{\cdot}\mathbf{x}, \quad p_\phi(\mathbf{x}, t) = \bar{p}_\phi(\mathbf{x} + \mathbf{X}(t), t) + \mathbf{A}(t)\boldsymbol{\cdot}\mathbf{x}$$

give the solution of the incompressible Euler equation in the space domain $\Omega = \mathbb{R}^3 \backslash B$ for the body frame, where p_ϕ is obtained from the Bernoulli equation and ϕ satisfies the Laplace equation $\triangle \phi = 0$ in Ω with the boundary conditions

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial B; \quad \phi \to -\mathbf{V}(t) \cdot \mathbf{x} \quad \text{as } |\mathbf{x}| \to \infty.$$

Problem 6. In this problem we derive a *generalized d'Alembert theorem* for the arbitrary translational motion of a solid body through an incompressible fluid at rest at infinity, working in the body frame as in part (b) of Problem 5.

(a) Following Lighthill (1979) we define a "pseudo-momentum" of the potential Euler solution by

$$\mathbf{P}_\phi = -\int_{\partial B} \phi \mathbf{n} \, dA.$$

Explain why the usual momentum $\int_\Omega \mathbf{u}_\phi \, dV$ diverges, but coincides with $\mathbf{P}_\phi$ up to an infinite constant and show that

$$\frac{d\mathbf{P}_\phi}{dt} = \int_{\partial B} \left(p_\phi + \frac{1}{2} |\mathbf{u}_\phi|^2 \right) \mathbf{n} \, dA.$$

(b) Prove that $\boldsymbol{\nabla}(\frac{1}{2}|\mathbf{u}_\phi|^2) = (\mathbf{u}_\phi \cdot \boldsymbol{\nabla}) \mathbf{u}_\phi$ and exploit this relation and the methods used to prove the d'Alembert theorem to show that

$$\int_{\partial B} \frac{1}{2} |\mathbf{u}_\phi|^2 \mathbf{n} \, dA = \mathbf{0}.$$

(c) Conclude from parts (a) and (b) that

$$\frac{d\mathbf{P}_\phi}{dt} = \int_{\partial B} p_\phi \mathbf{n} \, dA = \mathbf{F}_\phi$$

where $\mathbf{F}_\phi$ is the force of the body acting on the fluid. Conclude that the time-average

$$\overline{\mathbf{F}}_\phi := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{F}_\phi(t) \, dt = \mathbf{0}$$

whenever $\mathbf{V}(t)$ and thus $\mathbf{P}_\phi(t)$ remain bounded in time.