Homework #3 – Solutions

Problem 1. If A_0 is the cross-sectional area of the original vortex tube and if A is the same for the stretched tube, the conservation of vorticity flux (Helmholtz Theorem) implies

$$\omega_0 A_0 = \omega A.$$

On the other hand, conservation of volume (incompressibility) implies that

$$l_0 A_0 = l A.$$

Thus,

$$\frac{\omega}{\omega_0} = \frac{A_0}{A} = \frac{l}{l_0}. \qquad \underline{\text{QED}}$$

Problem 2. The equation for balance of momentum of the fluid is

$$\partial_t(\rho \mathbf{u}) + \rho \nabla\cdot(\mathbf{u}\mathbf{u} + p\mathbf{I} - 2\nu\mathbf{S}) = 0$$

Thus, the force of the body acting on the fluid $\mathbf{F}'$ is given by

$$\mathbf{F}' = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}\int_{\Omega} \rho\mathbf{u}\, dV$$

$$= -\int_{\partial\Omega} \rho\hat{\mathbf{n}}\cdot(\mathbf{u}\mathbf{u} + p\mathbf{I} - 2\nu\mathbf{S})\, dA \quad \text{by divergence theorem}$$

$$= -\rho\int_{\partial\Omega} (p\hat{\mathbf{n}} - 2\nu\mathbf{S}\cdot\hat{\mathbf{n}})\, dA \quad \text{using } \mathbf{u}\cdot\hat{\mathbf{n}} = 0$$

$$= -\rho\int_{\partial\Omega} (p\hat{\mathbf{n}} + \mathbf{T}_w)\, dA \quad \text{using } \mathbf{T}_w = -2\nu\mathbf{S}\cdot\hat{\mathbf{n}}$$

By Newton's third law, $\mathbf{F} = -\mathbf{F}' = \rho\int_{\partial B} (p\hat{\mathbf{n}} + \mathbf{T}_w)\, dA.$

Problem 3. (a) Note first by taking the gradient of

$$\frac{d}{dt} X^t(a) = u(X^t(a), t)$$

that

$$\frac{d}{dt} \nabla_a X^t(a) = \nabla_a X^t(a) \cdot \nabla_x u(X^t(a), t) \quad \text{by chain rule.}$$

In that case,

$$\frac{d}{dt} \nabla_a X^t(a) = -(\nabla_a X^t(a))^{-1} \frac{d}{dt} \nabla_a X^t(a) \cdot (\nabla_a X^t(a))^{-1}$$

$$= -\nabla_x u(X^t(a), t)(\nabla_a X^t(a))^{-1}.$$

With the definition of the Cauchy invariant $\Omega(a,t) = \omega(X^t(a),t) \cdot (\nabla_a X^t(a))^{-1}$ we obtain from the product rule

$$\frac{d}{dt} \Omega(a,t) = D_t \omega(X^t(a),t) \cdot (\nabla_a X^t(a))^{-1}$$

$$+ \omega(X^t(a),t) \cdot \frac{d}{dt} (\nabla_a X^t(a))^{-1}$$

$$= (\omega \cdot \nabla) u(X^t(a,t),t) \cdot (\nabla_a X^t(a))^{-1}$$

$$- \omega(X^t(a),t) \cdot \nabla_x u(X^t(a),t)(\nabla_a X^t(a))^{-1}$$

$$= 0.$$

(b) Starting from the definition of the Weber velocity

$$w(a,t) := \nabla_a X^t(a) \cdot v(a,t)$$

and the Euler equations written in Lagrangian form

$$\frac{\partial v}{\partial t}(a,t) = -\nabla p(X^t(a),t)$$

we obtain

$$\frac{\partial w}{\partial t}(a,t) = \nabla_a \frac{\partial X^t(a)}{\partial t} \cdot v(a,t) + \nabla_a X^t(a) \cdot \frac{\partial v}{\partial t}(a,t)$$

$$= \nabla_a v(a,t) \cdot v(a,t) - \nabla_a X^t(a) \cdot \nabla_x p(X^t(a),t)$$

$$= \nabla_a \left(\tfrac{1}{2} |v(a,t)|^2 - p(X^t(a),t) \right)$$

by the product rule and the chain rule, respectively.

(c) Again using the definition of w:

$$\oint_C da \cdot w(a,t) = \oint_C (da \cdot \nabla_a) X^t(a) \cdot u(X^t(a),t)$$

$$= \oint_{C(t)} dx \cdot u(x,t)$$

where $C(t) = X^t(C)$, by the change of variables formula. We then obtain from part (b) that

$$\frac{d}{dt} \oint_{C(t)} dx \cdot u(x,t) = \oint_C da \cdot \partial_t w(a,t)$$

$$= \oint_C da \cdot \nabla_a \left(\tfrac{1}{2} |v(a,t)|^2 - p(X^t(a),t) \right)$$

$$= 0$$

since the integrand of the line-integral is a total gradient.

d) Let us define $\Omega^*(a,t) = \nabla_a \times \mathbf{w}(a,t)$, or

$$\Omega_j^* = \epsilon_{jkl} \frac{\partial w_l}{\partial a_k}$$

$$= \epsilon_{jkl} \frac{\partial}{\partial a_k} \left[v_n(a,t) \frac{\partial X_n^t}{\partial a_l} \right] \quad \text{by definition of } \mathbf{w}$$

$$= \epsilon_{jkl} \left[\frac{\partial v_n}{\partial a_k} \frac{\partial X_n^t}{\partial a_l} + v_n \cancel{\frac{\partial^2 X_n^t}{\partial a_k \partial a_l}}^{\,0} \right] \quad \text{by symmetry in } k,l$$

$$= \epsilon_{jkl} \frac{\partial u_n}{\partial x_m} \frac{\partial X_m^t}{\partial a_k} \frac{\partial X_n^t}{\partial a_l} \quad \text{since } \frac{\partial v_n}{\partial a_k} = \frac{\partial u_n}{\partial x_m} \frac{\partial X_m^t}{\partial a_k}$$

by the chain rule

Since $\epsilon_{jkl} \frac{\partial X_m^t}{\partial a_k} \frac{\partial X_n^t}{\partial a_l}$ is anti-symmetric in m,n, only the anti-symmetric part Ω_{nm} of $u_{n,m} = \frac{\partial u_n}{\partial x_m}$ contributes, so that

$$\Omega_j^* = \epsilon_{jkl} \Omega_{nm} \frac{\partial X_m^t}{\partial a_k} \frac{\partial X_n^t}{\partial a_l}$$

$$= -\frac{1}{2} \epsilon_{jkl} \epsilon_{nmp} \omega_p \frac{\partial X_m^t}{\partial a_k} \frac{\partial X_n^t}{\partial a_l} \quad \text{using } \Omega_{nm} = -\frac{1}{2}\epsilon_{nmp}\omega_p.$$

Next, we calculate

$$\Omega_j^* \frac{\partial X_i^t}{\partial a_j} = -\frac{1}{2} \epsilon_{nmp} \omega_p \cdot \epsilon_{jkl} \frac{\partial X_i^t}{\partial a_j} \frac{\partial X_m^t}{\partial a_k} \frac{\partial X_n^t}{\partial a_l}$$

$$= -\frac{1}{2} \epsilon_{nmp} \omega_p \cdot \frac{\partial (X_i^t, X_m^t, X_n^t)}{\partial (a_1, a_2, a_3)}$$

$$= -\frac{1}{2} \epsilon_{nmp} \omega_p \cdot \epsilon_{imn}$$

since $\frac{\partial (X_1^t, X_2^t, X_3^t)}{\partial (a_1, a_2, a_3)} = 1$ and the determinant is anti-symmetric under interchange of columns. Thus,

$$\Omega_j^* \frac{\partial X_i^t}{\partial a_j} = \frac{1}{2} \epsilon_{imn} \epsilon_{pmn} \omega_p = \delta_{ip} \omega_p = \omega_i \quad \text{since } \epsilon_{imn}\epsilon_{pmn} = 2\delta_{ip}.$$

Thus, $$\Omega_j^* = \omega_i \left(\frac{\partial X^t}{\partial a} \right)^{-1}_{ij} = \Omega_j .$$

QED

Problem 4. (a) Since $\tilde{W}(s)$ does not depend upon x, we see by applying the gradient ∇_x to the equation

$$\hat{d}_s \tilde{A}_t^s(x) = u(\tilde{A}_t^s(x), s)\,ds + \sqrt{2\nu}\,\hat{d}\tilde{W}(s)$$

that

$$\hat{d}_s \nabla_x \tilde{A}_t^s(x) = \nabla_x \tilde{A}_t^s(x) \cdot \nabla_x u(\tilde{A}_t^s(x), s)\,ds,$$

which is an ODE without a stochastic term. Thus,

$$\frac{d}{ds}\left(\nabla_x \tilde{A}_t^s(x)\right)^{-1} = -\left(\nabla_x \tilde{A}_t^s(x)\right)^{-1} \frac{d}{ds}\nabla_x \tilde{A}_t^s(x)\left(\nabla_x \tilde{A}_t^s(x)\right)^{-1}$$

$$= -\nabla_x u(\tilde{A}_t^s(x), s) \cdot \left(\nabla_x \tilde{A}_t^s(x)\right)^{-1}.$$

Using the definition $\tilde{D}_t^s(x) = \left(\nabla_x \tilde{A}_t^s(x)\right)^{-T}$ we obtain by taking the transpose that

$$\frac{d}{ds}\tilde{D}_t^s(x) = -\tilde{D}_t^s(x)\left(\nabla_x u(\tilde{A}_t^s(x), s)\right)^T.$$

By its definition, $\tilde{A}_t^t(x) = x \Rightarrow \nabla_x \tilde{A}_t^t(x) = I \Rightarrow \tilde{D}_t^t(x) = I^{-T} = I$.

(b) Applying the backward Itō rule

$$\hat{d}_s \omega(\tilde{A}_t^s(x), s) = (\partial_s \omega - \nu\Delta\omega)(\tilde{A}_t^s(x), s)\,ds$$
$$+ \left(u(\tilde{A}_t^s(x), s)\,ds + \sqrt{2\nu}\,\hat{d}\tilde{W}(s)\right)\cdot\nabla\omega(\tilde{A}_t^s(x), s)$$

$$= (\omega\cdot\nabla)u(\tilde{A}_t^s(x), s)\,ds + \sqrt{2\nu}(\hat{d}\tilde{W}(s)\cdot\nabla)\omega(\tilde{A}_t^s(x), s)$$

using $D_s\omega(x,s) = (\omega\cdot\nabla)u(x,s)$ to obtain the second line.

Using then the definition

$$\tilde{\omega}_s(x,t) := \tilde{D}_t^s(x)\,\omega(\tilde{A}_t^s(x), s)$$

we obtain from the product rule

$$\hat{d}_s \tilde{\omega}_s(x,t) = \hat{d}_s \tilde{D}_t^s(x) \cdot \omega(\tilde{A}_t^s(x), s) + \tilde{D}_t^s(x) \cdot \hat{d}_s \omega(\tilde{A}_t^s(x), s)$$

$$= -\tilde{D}_t^s(x) \cdot (\nabla_x u(\tilde{A}_t^s(x), s))^{-T} \omega(\tilde{A}_t^s(x), s)\, ds$$

$$+ \tilde{D}_t^s(x) \cdot \Big[(\omega \cdot \nabla) u(\tilde{A}_t^s(x), s)\, ds + \sqrt{2\nu}\, (\hat{d}\tilde{W}(s) \cdot \nabla)\, \omega(\tilde{A}_t^s(x), s) \Big]$$

$$= \sqrt{2\nu}\, (\hat{d}\tilde{W}(s) \cdot \nabla)\, \omega(\tilde{A}_t^s(x), s),$$

as required. Here it is worth noting that the product rule in general must be modified for backward Itō calculus, and that generally

$$\hat{d}_s (fg) = (\hat{d}_s f) g + f(\hat{d}_s g) + \hat{d}_s \langle f, g \rangle$$

where $\langle f, g \rangle$ is the so-called <u>covariation</u> or <u>quadratic variation</u>. However, when one of f and g has finite variation (as $\tilde{D}_t^s(x)$ does above), then $\langle f, g \rangle \equiv 0$ and the usual product rule holds.

Finally, we note by integration that

$$\tilde{\omega}_s(x,t) = \omega(x,t) - \sqrt{2\nu} \int_s^t (\hat{d}\tilde{W}(r) \cdot \nabla)\, \omega(\tilde{A}_t^s(x), s)$$

and that the latter backward Itō integral is a <u>martingale backward in time</u> in the variable s.

Problem 5. Exactly as in Problem 4,

$$\hat{d}_s \omega(\tilde{A}^s_t(x), s) = (\partial_s \omega - \nu \Delta \omega)(\tilde{A}^s_t(x), s)\, ds$$

$$+ \left(u(\tilde{A}^s_t(x), s)\, ds + \sqrt{2\nu}\, \hat{d}\tilde{W}(s) - \nu n(\tilde{A}^s_t(x))\, \hat{d}\tilde{\ell}^s_t(x) \right) \cdot \nabla \omega(\tilde{A}^s_t(x), s)$$

$$= (\omega \cdot \nabla) u(\tilde{A}^s_t(x), s)\, ds + \left(\sqrt{2\nu}\, \hat{d}\tilde{W}(s) - \nu n(\tilde{A}^s_t(x))\, \hat{d}\tilde{\ell}^s_t(x) \right) \cdot \nabla \omega(\tilde{A}^s_t(x), s)$$

$$= (\omega \cdot \nabla) u(\tilde{A}^s_t(x), s)\, ds + \left(\sqrt{2\nu}\, \hat{d}\tilde{W}(s) \cdot \nabla \right) \omega(\tilde{A}^s_t(x), s)$$

$$+ \sigma^P(\tilde{A}^s_t(x), s)\, \hat{d}\tilde{\ell}^s_t(x).$$

It follows by the product rule that

$$\hat{d}_s \left(\tilde{D}^s_t(x)\, \omega(\tilde{A}^s_t(x), s) \right) = -\tilde{D}^s_t(x) \cdot \left(\nabla_x u(\tilde{A}^s_t(x), s) \right)^{-T} \omega(\tilde{A}^s_t(x), s)\, ds$$

$$+ \tilde{D}^s_t(x) \cdot \hat{d}_s \omega(\tilde{A}^s_t(x), s)$$

$$= \tilde{D}^s_t(x) \cdot \left[\left(\sqrt{2\nu}\, \hat{d}\tilde{W}(s) \cdot \nabla \right) \omega(\tilde{A}^s_t(x), s) + \sigma^P(\tilde{A}^s_t(x), s)\, \hat{d}\tilde{\ell}^s_t(x) \right]$$

On the other hand, with $\tilde{L}^s_t(x) := \int_s^t \tilde{D}^r_t(x) \cdot \sigma^P(\tilde{A}^r_t(x), r)\, \hat{d}\tilde{\ell}^r_t(x)$,

$$\hat{d}_s \tilde{L}^s_t(x) = -\tilde{D}^s_t(x) \cdot \sigma^P(\tilde{A}^s_t(x), s)\, \hat{d}\tilde{\ell}^s_t(x)$$

and therefore

$$\hat{d}_s \tilde{\omega}_s(x, t) = \hat{d}_s \left(\tilde{D}^s_t(x)\, \omega(\tilde{A}^s_t(x), s) + \tilde{L}^s_t(x) \right)$$

$$= \tilde{D}^s_t(x) \cdot \left(\sqrt{2\nu}\, \hat{d}\tilde{W}(s) \cdot \nabla \right) \omega(\tilde{A}^s_t(x), s),$$

as required.

Problem 6. (a) Using the incompressible Navier-Stokes equations in the form

$$\partial_t \mathbf{u} = \mathbf{u} \times \boldsymbol{\omega} - \nabla h - \nu \nabla \times \boldsymbol{\omega}$$

from the identity $-(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{u} \times (\nabla \times \mathbf{u}) - \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right)$, we can also rewrite this as

$$\partial_t u_i = \frac{1}{2}\epsilon_{ijk}\Sigma_{jk} - \partial_i h$$

with

$$\Sigma_{jk} = u_j\omega_k - u_k\omega_j - \nu\left(\frac{\partial\omega_k}{\partial x_j} - \frac{\partial\omega_j}{\partial x_k}\right)$$

$$= \epsilon_{jkl}\left(\mathbf{u}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega}\right)_l .$$

Taking the ensemble/time/xz-average, we obtain

$$0 = \frac{1}{2}\epsilon_{ijk}\overline{\Sigma}_{jk} - \partial_i\overline{h}$$

or, equivalently,

$$\overline{\Sigma}_{ij} = \epsilon_{ijk}\partial_k\overline{h}.$$

(i) $ij = yz$: From the above result

$$\overline{\Sigma}_{yz} = \frac{\partial\overline{h}}{\partial x}.$$

Also,

$$\overline{\Sigma}_{yz} = \overline{v\omega_z - w\omega_y} - \nu\frac{\partial\overline{\omega}_z}{\partial y}.$$

Here we have used the fact that $\overline{\omega}_z = -\partial\overline{u}/\partial y$ is the only non-vanishing component of mean vorticity. Finally,

$$\overline{v\omega_z - w\omega_y} - \nu \frac{\partial \overline{\omega_z}}{\partial y} = \overline{\Sigma}_{yz} = \frac{\partial \overline{h}}{\partial x} = \frac{\partial \overline{p}}{\partial x},$$

where in the last equality we used $h = p + \frac{1}{2}|\mathbf{u}|^2$ and $(\partial/\partial x)\,\overline{\frac{1}{2}|\mathbf{u}|^2} = 0$.

(ii) $\underline{ij = zx}$: Similarly, we obtain from the general relation

$$\overline{\Sigma}_{zx} = \frac{\partial \overline{h}}{\partial y}$$

and now

$$\overline{\Sigma}_{zx} = \overline{w\omega_x - u\omega_z}$$

since $\overline{\omega}_x = 0$ and $\partial \overline{\omega}_z / \partial x = 0$. Thus,

$$\overline{w\omega_x - u\omega_z} = \overline{\Sigma}_{zx} = \frac{\partial \overline{h}}{\partial y} = \frac{\partial \overline{p}}{\partial y} + \frac{\partial}{\partial y}\left(\frac{1}{2}\overline{u^2+v^2+w^2}\right)$$

from the definition of h.

(iii) $\underline{ij = xy}$: Again, since all z-derivatives vanish,

$$\overline{\Sigma}_{xy} = \frac{\partial \overline{h}}{\partial z} = 0$$

and

$$\overline{\Sigma}_{xy} = \overline{u\omega_y - v\omega_x}$$

since $\overline{\omega}_y = \overline{\omega}_x = 0$. Thus,

$$\overline{u\omega_y - v\omega_x} = \overline{\Sigma}_{xy} = 0.$$

(b) From part (a)

$$\frac{\partial}{\partial y}\left(\frac{\partial \bar{p}}{\partial x}\right) = \frac{\partial}{\partial y}\overline{\Sigma_{yz}}.$$

However, recall that

$$\partial_t w_z + \frac{\partial}{\partial x}\Sigma_{xz} + \frac{\partial}{\partial y}\Sigma_{yz} = 0.$$

Taking the average and noting that $\partial_t \overline{\omega_z} = \frac{\partial}{\partial x}\overline{\Sigma_{xz}} = 0$,

$$\frac{\partial}{\partial y}\left(\frac{\partial \bar{p}}{\partial x}\right) = \frac{\partial}{\partial y}\overline{\Sigma_{yz}} = 0.$$

(c) The general identity

$$\mathbf{u}\times\boldsymbol{\omega} = -\nabla\cdot(\mathbf{u}\mathbf{u}) + \nabla\left(\tfrac{1}{2}|\mathbf{u}|^2\right)$$

for x-component gives

$$\overline{v\omega_z - w\omega_y} = -\nabla\cdot(\overline{u\mathbf{u}}) + \cancelto{0}{\frac{\partial}{\partial x}\left(\tfrac{1}{2}\overline{|\mathbf{u}|^2}\right)}$$

$$= -\frac{\partial}{\partial y}\overline{uv} = -\frac{\partial}{\partial y}\overline{u'v'},$$

where $\overline{uv} = \overline{u'v'}$ since $\bar{v}=0$. Finally, the y-component of the vector calculus identity gives

$$\overline{w\omega_x - u\omega_z} = -\nabla\cdot(\overline{v\mathbf{u}}) + \frac{\partial}{\partial y}\left(\tfrac{1}{2}\overline{|\mathbf{u}|^2}\right)$$

$$= -\frac{\partial}{\partial y}\overline{v^2} + \frac{\partial}{\partial y}\left(\tfrac{1}{2}\overline{u^2+v^2+w^2}\right).$$