Homework No.3, 553.794, Due March 20, 2023.

Problem 1. Consider a vortex tube in the form of a right circular cylinder of length ℓ_0 with constant axial vorticity ω_0 across its cross sectional area. If this vortex tube is stretched axially into another right circular cylinder of length $\ell > \ell_0$ and axial vorticity ω, explain why $\omega/\omega_0 = \ell/\ell_0$ for an ideal incompressible fluid.

Problem 2. (a) Consider a solid body B immersed in an incompressible fluid with Newtonian viscous stress tensor $\mathbf{T}_\nu = -2\nu\mathbf{S}$, where $\mathbf{S}$ is the symmetric strain rate tensor. Using the momentum balance equation, explain why the force $\mathbf{F}$ that the fluid exerts on the body is given by the surface integral

$$\mathbf{F} = \rho \int_{\partial B} (p\hat{\mathbf{n}} + \boldsymbol{\tau}_w)\, dA,$$

where $\hat{\mathbf{n}}$ is the surface unit normal pointing into the body.

(b) Explain why Prandtl boundary layer theory suggests that C_D^f, the frictional contribution to the drag coefficient $C_D = F/\frac{1}{2}\rho U^2 A$ arising from the skin friction $\boldsymbol{\tau}_w$, should scale $\propto Re^{-1/2}$ with the Reynolds number.

Problem 3. (a) By taking the time derivative d/dt, show that the *Cauchy invariant*

$$\boldsymbol{\Omega}(\mathbf{a},t) := \boldsymbol{\omega}(\mathbf{X}^t(\mathbf{a}),t)\boldsymbol{\cdot}(\boldsymbol{\nabla}_a\mathbf{X}^t(\mathbf{a}))^{-1}$$

is a conserved quantity of 3D incompressible Euler for every particle label $\mathbf{a}$. Observe that for notational simplicity we have here suppressed explicit reference to the initial time t_0 in the Lagrangian flow map $\mathbf{X}_{t_0}^t(\mathbf{a})$.

(b) The *Weber velocity variable* is defined in terms of the standard Lagrangian velocity $\mathbf{v}(\mathbf{a},t) = d\mathbf{X}^t(\mathbf{a})/dt = \mathbf{u}(\mathbf{X}^t(\mathbf{a}),t)$ by

$$\mathbf{w}(\mathbf{a},t) := \boldsymbol{\nabla}_a\mathbf{X}^t(\mathbf{a})\boldsymbol{\cdot}\mathbf{v}(\mathbf{a},t)$$

and it is closely related to the Cauchy invariant. Establish the so-called Weber formulation of the 3D Euler equation:

$$\frac{\partial}{\partial t}\mathbf{w} = \boldsymbol{\nabla}_a\left[\frac{1}{2}|\mathbf{v}|^2 - p_L\right],$$

where note that $p_L(\mathbf{a},t) = p(\mathbf{X}^t(\mathbf{a}),t)$ is the Lagrangian pressure.

(c) If C is any fixed loop in the label space, show that

$$\oint_C d\mathbf{a}\boldsymbol{\cdot}\mathbf{w}(\mathbf{a},t) = \oint_{C(t)} d\mathbf{x}\boldsymbol{\cdot}\mathbf{u}(\mathbf{x},t)$$

where $C(t)$ is the image of C under the Lagrangian flow $\mathbf{X}^t(\mathbf{a})$. Then use the result in part (b) to give another proof of the Kelvin circulation theorem.

(d) Show that Cauchy's vorticity invariant is the curl of Weber's velocity variable:

$$\boldsymbol{\Omega}(\mathbf{a},t) = \boldsymbol{\nabla}_\alpha \times \mathbf{w}(\mathbf{a},t).$$

Hint: Define $\boldsymbol{\Omega}^*(\mathbf{a},t) \equiv \boldsymbol{\nabla}_a \times \mathbf{w}(\mathbf{a},t)$ and then calculate $\boldsymbol{\Omega}^*(\mathbf{a},t)\boldsymbol{\cdot}\boldsymbol{\nabla}_a \mathbf{X}^t(\mathbf{a})$. You will find useful the result

$$\epsilon_{ijk}\frac{\partial X_l}{\partial \alpha_i}\frac{\partial X_m}{\partial \alpha_j}\frac{\partial X_n}{\partial \alpha_k} = \epsilon_{lmn},$$

which you should show follows from the Jacobian, $\partial(X_1,X_2,X_3)/\partial(\alpha_1,\alpha_2,\alpha_3) = 1$.

Problem 4. We consider here the evolution equation backward in time for stochastic Lagrangian trajectories

$$\hat{d}_s \widetilde{\mathbf{A}}_t^s(\mathbf{x}) = \mathbf{u}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s)\,ds + \sqrt{2\nu}\,\hat{d}\widetilde{\mathbf{W}}(s), \quad s < t$$

where $\hat{d}_s$ denotes the backward stochastic Itō differential.

(a) Show that the *stochastic deformation matrix* $\widetilde{\mathbf{D}}_t^s(\mathbf{x}) := (\widetilde{\mathbf{A}}_t^s(\mathbf{x}))^{-\top}$ satisfies the ordinary differential equation

$$\frac{d}{ds}\widetilde{\mathbf{D}}_t^s(\mathbf{x}) = -\widetilde{\mathbf{D}}_t^s(\mathbf{x})(\boldsymbol{\nabla}_x \mathbf{u}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s))^\top, \quad s < t$$

and the final condition $\widetilde{\mathbf{D}}_t^t(\mathbf{x}) = \mathbf{I}$.

(b) If $\boldsymbol{\omega}(\mathbf{x},t)$ is the solution of the viscous Helmholtz equation, then use the result in part (a) to derive the following equation

$$\hat{d}_s \widetilde{\boldsymbol{\omega}}_s(\mathbf{x},t) = \sqrt{2\nu}\widetilde{\mathbf{D}}_t^s(\mathbf{x})\,(\hat{d}\widetilde{\mathbf{W}}(s)\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{\omega}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s), \quad s < t \qquad (*)$$

for the *stochastic Cauchy invariant* $\widetilde{\boldsymbol{\omega}}_s(\mathbf{x},t) := \widetilde{\mathbf{D}}_t^s(\mathbf{x})\boldsymbol{\omega}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s)$. You will need to use the result for any smooth function $f(\mathbf{x},t)$ that

$$\hat{d}_s f(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s) = (\partial_s f - \nu\triangle f)(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s)ds + (\hat{d}_s\widetilde{\mathbf{A}}_t^s(\mathbf{x})\boldsymbol{\cdot}\boldsymbol{\nabla})f(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s)$$

This is the *backward Itō rule*, which is the replacement of the standard chain rule for the backward Itō differential.

Problem 5. We now repeat the previous problem for stochastic Lagrangian trajectories in a domain Ω which are *reflected at the boundary* $\partial\Omega$. These satisfy

$$\hat{d}_s\widetilde{\mathbf{A}}_t^s(\mathbf{x}) = \mathbf{u}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s)\,ds + \sqrt{2\nu}\hat{d}\widetilde{\mathbf{W}}(s) - \nu\mathbf{n}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}))\hat{d}\widetilde{\ell}_t^s(\mathbf{x}), \quad s<t$$

where $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}\in\partial\Omega$ is the surface normal vector pointing into the domain. Note that we have defined the (backward in time) boundary local-time density $\widetilde{\ell}_t^s(\mathbf{x})$ so that it *decreases* each time that the trajectory $\widetilde{\mathbf{A}}_t^s(\mathbf{x})$ hits the boundary. If we now *define* the deformation matrix by the ODE in part (a) of Problem 4, then prove that the modified stochastic Cauchy invariant

$$\widetilde{\boldsymbol{\omega}}_s(\mathbf{x},t) := \widetilde{\mathbf{D}}_t^s(\mathbf{x})\boldsymbol{\omega}(\widetilde{\mathbf{A}}_t^s(\mathbf{x}),s) + \int_s^t \widetilde{\mathbf{D}}_t^r(\mathbf{x})\boldsymbol{\cdot}\boldsymbol{\sigma}^P(\widetilde{\mathbf{A}}_t^r(\mathbf{x}),r)\,\hat{d}\widetilde{\ell}_t^r(\mathbf{x}),$$

with $\boldsymbol{\sigma}^P(\mathbf{x},t) := -\nu(\mathbf{n}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{\omega}(\mathbf{x},t)$ satisfies equation (*) in part (b) of Problem 4.

Problem 6. We consider some statistical relations for *turbulent channel flow* between two plane-parallel walls and driven by an applied pressure gradient (Poiseuille flow). Here we take x to be the streamwise direction along the pressure gradient, y the direction normal to the walls, and z the third spanwise direction, with the flow velocity assumed to satisfy periodic b.c. in the x- and z-directions. With these assumptions, all long-time averages $\bar{a}$ of flow quantities a are constant in x and z, except for the pressure field p which has $\partial\bar{p}/\partial x<0$. Note also that the only non-vanishing component of the mean velocity is $\bar{u}$ in the streamwise direction, with wall-normal component $\bar{v}=0$ and spanwise component $\bar{w}=0$.

(a) Prove the following three relations

$$\begin{aligned}\overline{v\omega_z - w\omega_y} - \nu\tfrac{\partial\bar{\omega}_z}{\partial y} &= \overline{\Sigma}_{yz} = \tfrac{\partial\bar{h}}{\partial x} = \tfrac{\partial\bar{p}}{\partial x}\\ \overline{w\omega_x - u\omega_z} &= \overline{\Sigma}_{zx} = \tfrac{\partial\bar{h}}{\partial y}\\ \overline{u\omega_y - v\omega_v} &= \overline{\Sigma}_{zx} = 0\end{aligned}$$

where $h = p + \frac{1}{2}|\mathbf{u}|^2$ is the total pressure (hydrostatic plus dynamic).

Hint: Rewrite the Navier-Stokes equation as $\partial_t u_i = \frac{1}{2}\epsilon_{ijk}\Sigma_{jk} - \partial_i h$ where ϵ_{ijk} is the anti-symmetric Levi-Civita tensor.

(b) Use the first result in part (a) to prove that $\frac{\partial}{\partial y}\left(\frac{\partial\bar{p}}{\partial x}\right)=0$.

(c) Derive the following kinematic identities

$$\begin{aligned}\overline{v\omega_z - w\omega_y} &= -\frac{\partial}{\partial y}\overline{uv} = -\frac{\partial}{\partial y}\overline{u'v'}\\ \overline{w\omega_x - u\omega_z} &= -\frac{\partial}{\partial y}\overline{v^2} + \frac{\partial}{\partial y}\overline{u^2+v^2+w^2}.\end{aligned}$$