Homework #2
Solutions

Problem 1. Following the hint, we write

$$\boldsymbol{\nabla} = \mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n}$$

near the boundary $C = \partial\Omega$. Note first that

$$\boldsymbol{\nabla}\mathbf{u}\Big|_C = \mathbf{t}\frac{\partial \mathbf{u}}{\partial s} + \mathbf{n}\frac{\partial \mathbf{u}}{\partial n}\Big|_C = \mathbf{n}\epsilon_w$$

since $\mathbf{u} \equiv 0$ on C and thus $\partial \mathbf{u}/\partial s \equiv 0$ as well. For the Laplacian $\Delta \mathbf{u}$ we can likewise calculate

$$\Delta \mathbf{u} = \left(\mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n}\right)\cdot\left(\mathbf{t}\frac{\partial \mathbf{u}}{\partial s} + \mathbf{n}\frac{\partial \mathbf{u}}{\partial n}\right)$$

$$= \left(\frac{\partial^2 \mathbf{u}}{\partial s^2} + \mathbf{t}\cdot\frac{\partial \mathbf{t}}{\partial s}\frac{\partial \mathbf{u}}{\partial s} + \mathbf{n}\cdot\frac{\partial \mathbf{t}}{\partial n}\frac{\partial \mathbf{u}}{\partial s}\right)$$

$$+ \left(\mathbf{t}\cdot\frac{\partial \mathbf{n}}{\partial s}\frac{\partial \mathbf{u}}{\partial n} + \frac{\partial^2 \mathbf{u}}{\partial n^2}\right)$$

where the first parenthetical expression arises from the divergence of $\mathbf{t}\,\partial \mathbf{u}/\partial s$ and the second expression arises from $\mathbf{n}\,\partial \mathbf{u}/\partial n$. We have made use of $\mathbf{t}\cdot\mathbf{n} = 0$ to eliminate some terms. If we now consider the restriction to the wall, then

$$\Delta \mathbf{u}\Big|_C = \mathbf{t}\cdot\frac{\partial \mathbf{n}}{\partial s}\frac{\partial \mathbf{u}}{\partial n} + \frac{\partial^2 \mathbf{u}}{\partial n^2}\Big|_C$$

since again $\partial \mathbf{u}/\partial s = \partial^2 \mathbf{u}/\partial s^2 \equiv 0$ on C.

Recalling also the relation

$$\mathbf{t} \cdot \frac{\partial \mathbf{n}}{\partial s} = -\mathbf{n} \cdot \frac{\partial \mathbf{t}}{\partial s}$$

obtained by differentiating $\mathbf{t} \cdot \mathbf{n} = 0$, we then use the result from the hint

$$\frac{\partial \mathbf{t}}{\partial s} = \pm \kappa \mathbf{n}$$

which follows from the Frenet-Serret equations and our definition of $\mathbf{n}$ as pointing inward to Ω. Thus,

$$\Delta \mathbf{u}\Big|_C = \mp \kappa \frac{\partial \mathbf{u}}{\partial n} + \frac{\partial^2 \mathbf{u}}{\partial n^2}\Big|_C .$$

and using the Navier-Stokes equation on the boundary

$$\nu \Delta \mathbf{u} - \nabla p = \mathbf{0} \quad \text{on } C,$$

we obtain

$$\nu \frac{\partial^2 \mathbf{u}}{\partial n^2} = \pm \kappa \boldsymbol{\tau}_w + \nabla p \quad \text{on } C.$$

We have set $\boldsymbol{\tau}_w = \nu \boldsymbol{\varepsilon}_w$. Now taking the tangential component

$$\nu \frac{\partial^2 u_t}{\partial n^2} = \pm \kappa \tau_w + \frac{\partial p}{\partial s}$$

whereas the normal component gives

$$\nu \frac{\partial^2 u_n}{\partial n^2} = \frac{\partial p}{\partial n},$$

since $\mathbf{n} \cdot \boldsymbol{\tau}_w = 0$.

Problem 2. (a) From the defining equations

$$\dot{x} = u, \quad \dot{y} = v$$

we obtain

$$\dot{x}_\alpha = u_x x_\alpha + u_y y_\alpha, \quad \alpha = \xi, \eta$$

$$\dot{y}_\alpha = v_x x_\alpha + v_y y_\alpha, \quad \alpha = \xi, \eta$$

by taking derivatives of both sides with respect to either ξ or η.

In that case,

$$\frac{d}{dt}(x_\xi y_\eta - x_\eta y_\xi)$$

$$= (u_x x_\xi + \cancel{u_y y_\xi}) y_\eta + x_\xi (\cancel{v_x x_\eta} + v_y y_\eta)$$

$$- (u_x x_\eta + \cancel{u_y y_\eta}) y_\xi - x_\eta (\cancel{v_x x_\xi} + v_y y_\xi)$$

$$= (u_x + v_y)(x_\xi y_\eta - x_\eta v_y)$$

Note that at the initial time $x_\xi = y_\eta = 1$, $x_\eta = y_\xi = 0$

so that

$$x_\xi y_\eta - x_\eta y_\xi \big|_{t=0} = 1.$$

Since this linear equation for $(x_\xi y_\eta - x_\eta y_\xi)$ has a unique solution, it follows that

$$u_x + v_y \equiv 0 \quad \text{iff} \quad x_\xi y_\eta - x_\eta y_\xi \equiv 1.$$

(b) Note that

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}^{-1}$$

since $(\xi(x,y), \eta(x,y))$ is the inverse map to $(x(\xi,\eta), y(\xi,\eta))$.

However, because

$$\begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = x_\xi y_\eta - x_\eta y_\xi \equiv 1$$

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}$$

and thus we read off

$$\xi_x = y_\eta, \quad \xi_y = -x_\eta$$
$$\eta_x = -y_\xi, \quad \eta_y = x_\xi .$$

(c) We note that by the chain rule and the equation $\dot{x} = u$

$$\begin{aligned} \ddot{x} &= \frac{d}{dt} u(x,y,t) \\ &= u_t + \dot{x} u_x + \dot{y} u_y \\ &= u_t + u u_x + v u_y \end{aligned}$$

Next, we use the x-momentum equation of the Prandtl theory to infer that

$$\ddot{x} = -p_x + \partial_y^2 u$$

We next use

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$$

$$= -x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta}$$

by exploiting the relations from part (b). However,

$$u = \dot{x}$$

and

$$\frac{\partial^2}{\partial y^2} = \left(x_\eta \partial_\xi - x_\xi \partial_\eta\right)^2$$

so that we obtain the final result that

$$\ddot{x} = -p_x(x) + \left(x_\eta \partial_\xi - x_\xi \partial_\eta\right)^2 \dot{x}.$$

This is a second-order in time, nonlinear PDE for the Lagrangian flow map $x(\xi, \eta)$. The other component $y(\xi, \eta)$ can be obtained from the relation $x_\xi y_\eta - x_\eta y_\xi \equiv 1$.

QED

Problem 3. (a) First we note that from $\mathbf{n} = \nabla d(\mathbf{x})$

$$\frac{\partial \mathbf{n}}{\partial n} := (\mathbf{n}\cdot\nabla)\mathbf{n} = \mathbf{n}\cdot(\nabla\nabla d)$$
$$= n_i \nabla n_i \qquad \text{(summing over repeated } i\text{)}$$
$$= \nabla\left(\tfrac{1}{2} n_i n_i\right)$$
$$= 0$$

since $\mathbf{n}\cdot\mathbf{n} = 1$. Then using the above

$$\mathbf{n}\cdot\boldsymbol{\epsilon}_w = \mathbf{n}\cdot\frac{\partial \mathbf{u}}{\partial n}\Big|_S = \frac{\partial}{\partial n}(\mathbf{n}\cdot\mathbf{u})\Big|_S = 0$$

since by incompressibility and stick b.c.

$$\frac{\partial u_n}{\partial n} = -\nabla_s\cdot\mathbf{u}_s \equiv 0 \quad \text{on } S.$$

(b) We begin with

$$\boldsymbol{\tau}_w = 2\nu\mathbf{S}\cdot\mathbf{n} = \nu\left(\frac{\partial \mathbf{u}}{\partial n} + \nabla u_j\, n_j\right)\Big|_S$$
$$= \nu\frac{\partial \mathbf{u}}{\partial n} + \nu\nabla(u_j n_j)\Big|_S$$

since $u_j \nabla n_j \equiv 0$ on S. However, by part (a)

$$\frac{\partial}{\partial n}(u_j n_j) = \frac{\partial u_n}{\partial n} = 0$$

and

$$\nabla_s u_n \equiv 0 \quad \text{on } S$$

because of the stick b.c. Thus, we see that

$$\tau_w = \nu \frac{\partial u}{\partial n} = \nu \epsilon_w \quad \text{on } S.$$

For the second relation we perform a similar calculation

$$(\omega \times n)_i = \epsilon_{ijk}\, \omega_j\, n_k$$

$$= \epsilon_{ijk}\, \epsilon_{jlm}\, \partial_l u_m\, n_k$$

$$= (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km})\, \partial_l u_m\, n_k$$

$$= (n_k \partial_k) u_i - (\partial_i u_k)\, n_k$$

$$= \frac{\partial u_i}{\partial n} - 0 \qquad \text{since } (\partial_i u_k) n_k = \partial_i (u_k n_k) = 0 \text{ as above}$$

or

$$\omega \times n = \frac{\partial u}{\partial n} = \epsilon_w .$$

Thus,

$$\tau_w = \nu \epsilon_w = \nu\, \omega \times n .$$

(c) We note that

$$n_i K_{ij} = -\frac{\partial n_j}{\partial n} = 0$$

from part (a), while

$$K_{ij} n_j = -(\partial_i n_j) n_j = -\partial_i \left(\tfrac{1}{2} n_j n_j\right) = 0$$

since $n_j n_j \equiv 1$.

(d) Incompressibility implies that

$$\frac{\partial u_n}{\partial n} + \nabla_s \cdot \mathbf{u}_s = 0$$

Taking a second derivative $\partial/\partial n$ then gives

$$-\frac{\partial^2 u_n}{\partial n^2} = \nabla_s \cdot \frac{\partial \mathbf{u}_s}{\partial n} = \nabla_s \cdot \boldsymbol{\epsilon}_w \quad \text{on } S.$$

Next we use $\boldsymbol{\epsilon}_w = \boldsymbol{\omega}_w \times \mathbf{n}$ from part (b) and a vector calculus identity to obtain

$$\begin{aligned}\nabla_s \cdot \boldsymbol{\epsilon}_w &= \nabla_s \cdot (\boldsymbol{\omega}_w \times \mathbf{n}) \\ &= (\nabla_s \times \boldsymbol{\omega}_w) \cdot \mathbf{n} - \boldsymbol{\omega}_w \cdot (\nabla \times \mathbf{n})\end{aligned}$$

Note that

$$\boldsymbol{\omega}_w \cdot (\nabla \times \mathbf{n}) = \epsilon_{ijk}\, \omega_i\, \partial_j n_k = -\epsilon_{ijk}\, \omega_i\, K_{jk}.$$

However, $\boldsymbol{\omega}$ is tangent to S and can thus be written as $\boldsymbol{\omega} = \alpha \mathbf{e} + \beta \mathbf{f}$ where $\mathbf{e}, \mathbf{f}$ is a basis for the tangent space. Likewise, from part (c) we see that

$$\nabla \mathbf{n} = \gamma\, \mathbf{e}\mathbf{e} + \delta\, \mathbf{e}\mathbf{f} + \epsilon\, \mathbf{f}\mathbf{e} + \theta\, \mathbf{f}\mathbf{f}$$

so that $\epsilon_{ijk}\, \omega_i\, \partial_j n_k \equiv 0$. It follows that

$$\nabla_s \cdot \boldsymbol{\epsilon}_w = (\nabla_s \times \boldsymbol{\omega}_w) \cdot \mathbf{n} = (\mathbf{n} \times \nabla_s) \cdot \boldsymbol{\omega}_w.$$

QED

Problem 4. (a) The linearization at the fixed point $A = \nabla u(x_*)$ is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

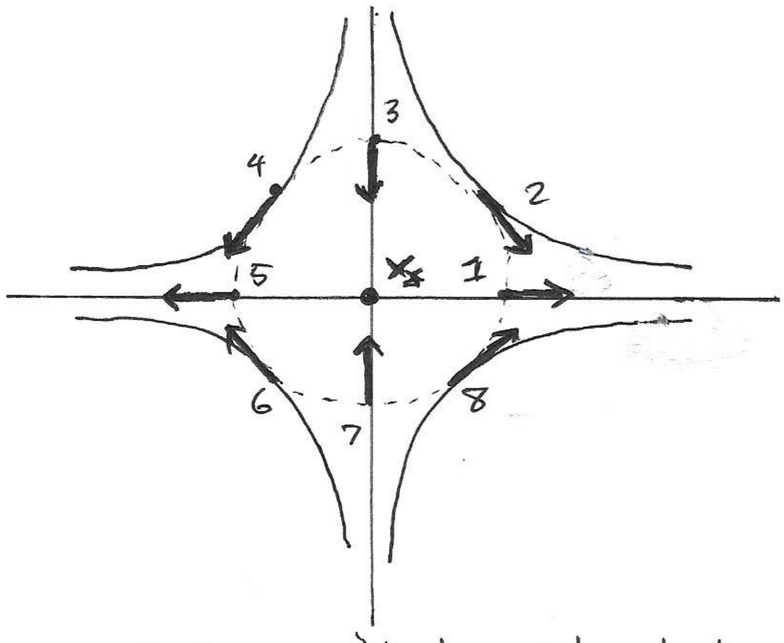
which has $\det A = -1 < 0$ and is thus a saddle point.

In fact, the y-axis is the stable manifold and the x-axis is the unstable manifold. Note also that

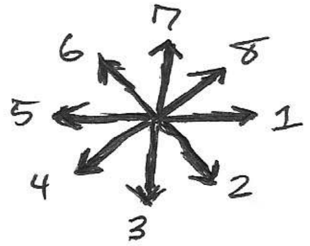
$$\frac{d}{dt}(xy) = \dot{x}y + x\dot{y} = x \cdot y + x \cdot (-y) = 0$$

so that phase orbits are given by hyperbolae with $xy = \text{const}$.

The phase portrait looks like:



On a circle around the critical point at the origin we indicate the unit vectors $\hat{u}$ at eight points numbered counterclockwise. These are the vectors



Clearly, these vectors rotate once clockwise so that

$$\text{Index} = -1.$$

(b) The linearization at the fixed point is

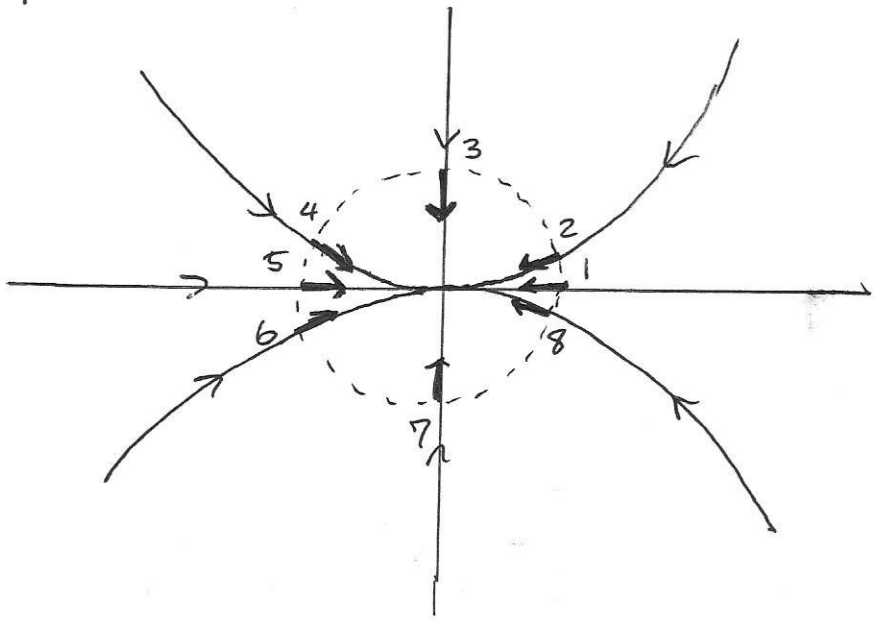
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

which has $\det A = 2 > 0$, $\operatorname{tr} A = -3 < 0$, $\operatorname{Dis} A = 9^2 - 8 = 1 > 0$ so that the critical point is a stable node. Note that

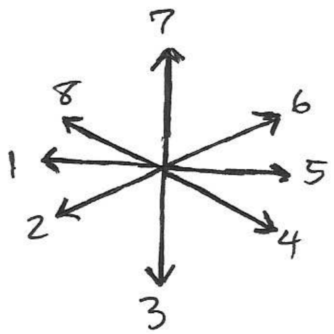
$$\frac{d}{dt}\left(\frac{y}{x^2}\right) = \frac{y}{x^3}(\dot{y}x - 2y\dot{x})$$
$$= \frac{y}{x^3}(-2yx + 2yx) = 0$$

so that phase orbits are parabolae $y = Cx^2$, $C = \text{const}$.

The phase portrait is:



The vector plot is now



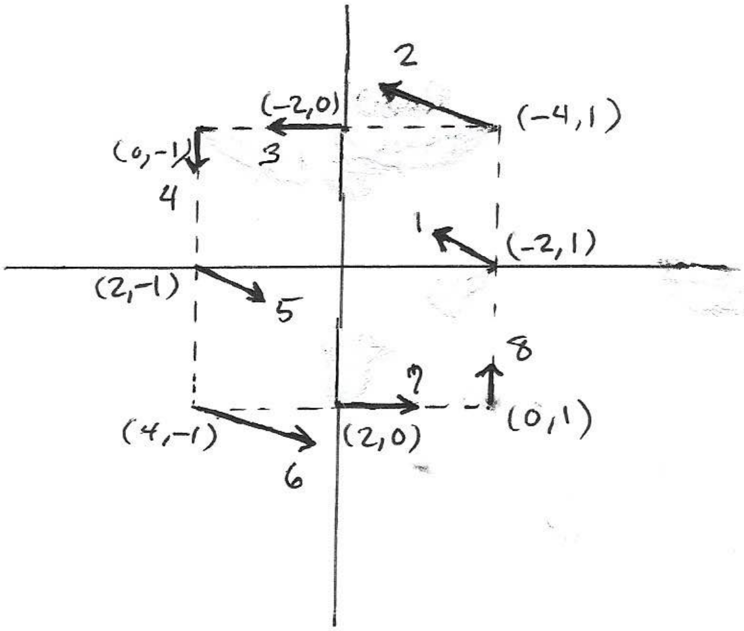
So that $\hat{u}$ rotates <u>once counterclockwise</u> and thus

$$\text{Index} = +1$$

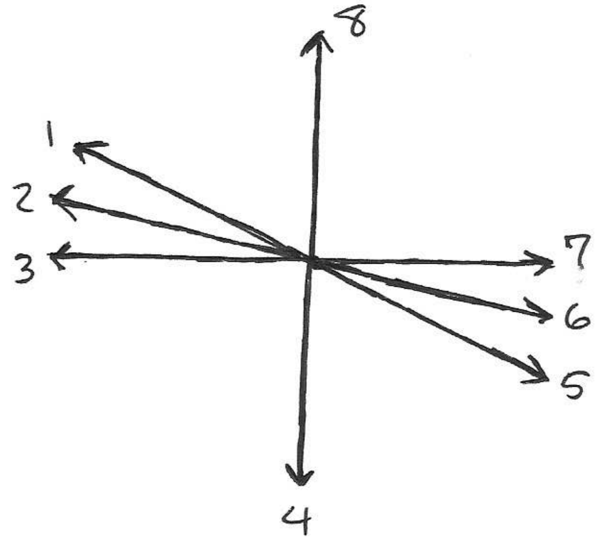
(c) The linearization at the fixed point is

$$\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$$

so that $\det \mathbf{A} = 2 > 0$, $\operatorname{tr} \mathbf{A} = -2 < 0$, $\operatorname{dis} \mathbf{A} = 4 - 8 = -4 < 0$ and thus the critical point is a <u>stable spiral</u>. It is easiest here to consider a square box around the critical point



which yields the vector plot



so that again $\hat{\mathbf{u}}$ rotates once counterclockwise and thus

$$\text{Index} = +1 .$$

Note that the continuity of the index under changes of the contour C can be inferred from the integral expression

$$\text{Index}(\mathbf{f}, \mathbf{x}_*) = \frac{1}{2\pi} \oint_C \frac{\mathbf{f} \times d\mathbf{f}}{\|\mathbf{f}\|^2} .$$

Since the index is continuous in C but also integer-valued, it does not change under deformations of C, as long as these do not hit the critical point $\mathbf{x}_*$. Thus, the value is independent of the particular contour C selected to encircle the critical point.

Problem 5. (a) Note by change of variable $s \to \bar{y} = sy$ that

$$\bar{u}_y(x,y,t) := \int_0^1 u_y(x, sy, t)\, ds$$

$$= \frac{1}{y} \int_0^y u_y(x, \bar{y}, t)\, d\bar{y}$$

$$= \frac{1}{y}\left[u(x,y,t) - u(x,0,t) \right]$$

$$= \frac{1}{y} u(x,y,t) \qquad \text{since } u(x,0,t) = 0$$

and thus

$$u(x,y,t) = \bar{u}_y(x,y,t)\, y .$$

Similarly, we change the variable $s \to \bar{y} = rys$ to obtain

$$\int_0^1 v_{yy}(x, rsy, t)\, r\, ds$$

$$= \frac{1}{y} \int_0^{ry} v_y(x, \bar{y}, t)\, d\bar{y}$$

$$= \frac{1}{y}\left[v_y(x, ry, t) - v_y(x, 0, t) \right]$$

$$= \frac{1}{y} v_y(x, ry, t)$$

since $v_y(x,0,t) = -u_x(x,0,t) = 0$ by incompressibility and stick b.c.

In that case, using the definition of $\overline{\overline{v}}_{yy}(x,y,t)$

$$\int_0^1 \int_0^1 v_{yy}(x, rsy, t)\, 2r\, dr\, ds$$

$$= \frac{2}{y} \int_0^1 v_y(x, ry, t)\, dr \qquad r \to \bar{y} = ry$$

$$= \frac{2}{y^2} \int_0^y v_y(x, \bar{y}, t)\, d\bar{y}$$

$$= \frac{2}{y^2} \left[v(x,y,t) - v(x,0,t) \right]$$

$$= \frac{2}{y^2} v(x,y,t)$$

and thus

$$v(x,y,t) = \frac{1}{2} \overline{\overline{v}}_{yy}(x,y,t) y^2.$$

(b) By incompressibility

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(\bar{u}_y y) + \frac{\partial}{\partial y}\left(\frac{1}{2}\overline{\overline{v}}_{yy} y^2\right)$$

$$= y \cdot \bar{u}_{y,x} + \frac{1}{2} y^2 \cdot \overline{\overline{v}}_{yy,y} + y \cdot \overline{\overline{v}}_{yy}$$

$$\Longrightarrow \qquad \bar{u}_{y,x} + \overline{\overline{v}}_{yy} + \frac{1}{2}\overline{\overline{v}}_{yy,y} = 0.$$

(c) Finally, $x = \gamma + y F(y,t)$ is a material curve if and only if each point of it evolves according to the equations

$$\dot{x} = u = \bar{u}_y\, y$$
$$\dot{y} = v = \frac{1}{2} \bar{\bar{v}}_{yy}\, y^2$$

Substituting into the first equation yields by chain rule

$$y F_t(y,t) + \dot{y} F(y,t) + y \dot{y} F_y(y,t)$$
$$= \bar{u}_y(x, y, t)\, y$$
$$= \bar{u}_y(\gamma + yF, y, t)\, y$$

so that

$$F_t(y,t) = \bar{u}_y(\gamma + yF, y, t) - \frac{\dot{y}}{y}\left(F + yF_y\right).$$

Now substituting from the second equation for $\dot{y}$ gives

$$F_t(y,t) = \bar{u}_y(\gamma + yF, y, t)$$
$$- \frac{1}{2} y\, \bar{\bar{v}}_{yy}(\gamma + yF, y, t)(F + yF_y)$$

<u>QED</u>

Problem 6, (a) The vortex at $(x,y)=(0,a)$ has velocity

$$u_+ = \frac{\kappa(y-a)}{x^2+(y-a)^2}, \quad v_+ = \frac{-\kappa x}{x^2+(y-a)^2}$$

while the vortex at $(x,y)=(0,-a)$ has velocity

$$u_- = \frac{-\kappa(y+a)}{x^2+(y+a)^2}, \quad v_- = \frac{\kappa x}{x^2+(y+a)^2}.$$

The vortex at $(x,y)=(0,a)$ feels the velocity of the other vortex, which is

$$u_-(0,a) = \frac{-\kappa}{2a}, \quad v_-(0,a)=0$$

On the other hand, the vortex at $(x,y)=(0,-a)$ feels the velocity of the first vortex, which is

$$u_+(0,-a) = -\frac{\kappa}{2a}, \quad v_+(0,-a)=0.$$

Thus, the pair of vortices are mutually advected with the velocities

$$u_{vort} = -\frac{\kappa}{2a}, \quad v_{vort} = 0$$

and move parallel to the x-axis together.

(b) When the point vortices are located at $(x,y)=(u_{vort}t, \pm a)$, then the resultant velocity at all other points is given by

$$u = \frac{\kappa(y-a)}{(x-u_{vort}t)^2+(y-a)^2} - \frac{\kappa(y+a)}{(x-u_{vort}t)^2+(y+a)^2}$$

$$v = \frac{-\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+(y-a)^2} + \frac{\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+(y+a)^2}$$

In particular, at $y=0$

$$v(x,0,t) = \frac{-\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+a^2} + \frac{\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+a^2}$$

$$\equiv 0.$$

Thus, we see that the velocity in the upper half-plane satisfies the no-flow-through condition at $y=0$. The vortex in the lower half-plane is the so-called <u>image vortex</u> required to enforce that condition for the flow in the upper half-plane.

(c) Since vortices are material points in 2D, adding the uniform velocity U changes the velocity of both vortices to

$$u_{vort} = U - \frac{\kappa}{2a}$$

Using this new definition of u_{vort}, we then obtain

$$u = U + \frac{\kappa(y-a)}{(x-u_{vort}t)^2+(y-a)^2} - \frac{\kappa(y+a)}{(x-u_{vort}t)^2+(y+a)^2}$$

$$v = \frac{-\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+(y-a)^2} + \frac{\kappa(x-u_{vort}t)}{(x-u_{vort}t)^2+(y+a)^2}.$$

(d) The streamwise velocity on the boundary $y=0$ now becomes

$$u = U - \frac{2\kappa a}{(x-u_{vort}t)^2+a^2}$$

Note that

$$\alpha = \frac{u_{vort}}{U} = 1 - \frac{\kappa}{2aU}$$

$$\Longrightarrow \kappa = 2aU(1-\alpha)$$

Thus,

$$\frac{u}{U} = 1 - \frac{4a^2(1-\alpha)}{(x-u_{vort}t)^2+a^2}, \quad u_{vort} = \alpha U.$$

As long as $\alpha < \frac{3}{4}$, then $a^2(1-\alpha) > 0$ and we see that the most negative value is achieved for $x = u_{vort}t$ where

$$\frac{|u|_{max}}{U} = 4(1-\alpha) - 1 > 0.$$

Problem 7. (a) Noting that $U^2 = \mathbf{U} \cdot \mathbf{U}$, we see that

$$\nabla\left(\tfrac{1}{2}U^2\right) = U_j \nabla U_j \qquad \text{(sum over repeated } i\text{)}$$

However,

$$\partial_i U_j = \partial_i(\partial_j \Phi) = \partial_j(\partial_i \Phi) = \partial_j U_i$$

is symmetric in i, j (i.e. there is no vorticity for potential flow!). Thus

$$U_j \partial_i U_j = U_j \partial_j U_i = (\mathbf{U} \cdot \nabla) U_i$$

and

$$\nabla\left(\tfrac{1}{2}U^2\right) = (\mathbf{U} \cdot \nabla)\mathbf{U}.$$

(b) From the Bernoulli equation we obtain

$$-\nabla P = \nabla\left(\tfrac{1}{2}U^2\right) = (\mathbf{U} \cdot \nabla)\mathbf{U}$$

by part (a). This, of course, is just the stationary Euler equation! Now substituting $\mathbf{U} = U\mathbf{t}_s$ and noting that $\mathbf{t}_s \cdot \nabla = \frac{\partial}{\partial s}$ gives

$$\begin{aligned} -\nabla P &= U \frac{\partial}{\partial s}(U \mathbf{t}_s) \\ &= U \frac{\partial U}{\partial s}\mathbf{t}_s + U^2 \frac{\partial \mathbf{t}_s}{\partial s} \end{aligned}$$

Now, using the Frenet - Serret equation

$$\frac{\partial \mathbf{t}_s}{\partial s} = \kappa_s \mathbf{n}_s$$

and $U \frac{\partial U}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right)$, we get finally that

$$-\nabla P = \frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right)\mathbf{t}_s + U^2 \kappa_s \mathbf{n}_s .$$

(c) Assuming continuity of the pressure-gradient across the thin boundary layer, we calculate the Lighthill source at the wall surface as

$$\boldsymbol{\sigma} = -\mathbf{n} \times \nabla P = \frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right)\mathbf{n} \times \mathbf{t}_s + U^2 \kappa_s \mathbf{n} \times \mathbf{n}_s$$

Since $\mathbf{t}_s$ is, by definition, the streamwise direction, then $\mathbf{n} \times \mathbf{t}_s$ is the spanwise direction and the term $\propto \frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right)$ creates spanwise vorticity. On the other hand, if the streamlines bend in a direction perpendicular to both $\mathbf{n}$ and $\mathbf{t}_s$, e.g. to pass around an obstacle in the flow, then

$$\boldsymbol{\sigma} \cdot \mathbf{t}_s = U^2 \kappa_s (\mathbf{n} \times \mathbf{n}_s) \cdot \mathbf{t}_s \neq 0$$

and the term $\propto U^2 \kappa_s$ can thus generate streamwise vorticity.