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# Remarks on Zero Viscosity Limit for Nonstationary Navier-Stokes Flows with Boundary

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## 1. Introduction

This paper is concerned with the question of convergence of the nonstationary, incompressible Navier-Stokes flow $u = u_\nu$ to the Euler flow $\bar{u}$ as the viscosity $\nu$ tends to zero. If the underlying space domain is all of $R^m$, the convergence has been proved by several authors under appropriate assumptions on the convergence of the data (initial condition and external force); see Golovkin [1] and McGrath [2] for $m = 2$ and all time, and Swann [3] and the author [4,5] for $m = 3$ and short time. The case $m \geqslant 4$ can be handled in the same way; in fact, the simple method given in [5] applies to any dimension. All these results refer to strong solutions (or even classical solutions, depending on the data) of the Navier-Stokes equation.

The problem becomes extremely difficult if the space domain $\Omega \subset R^m$ has nonempty boundary $\partial\Omega$, due to the appearance of the boundary layer, and remains open as far as the author is aware. Here it is necessary, in general, to consider weak solutions u of the Navier-Stokes equation, since strong solutions are known to exist only for a short time interval that tends to zero as $\nu \to 0$ (except for $m = 2$), while weak solutions are known to exist for all time for any initial data in $L^2(\Omega)$, although their uniqueness is not known.

The purpose of this paper is to give some necessary and sufficient conditions for the convergence to take place. In particular, we shall show that, roughly speaking, $u \to \bar{u}$ in $L^2(\Omega)$, uniformly in $t \in [0,T]$, if and only if the energy dissipation for u during the interval [0,T] tends to zero. Here [0,T] is an interval on which the smooth solution $\bar{u}$ of the Euler equation exists, and u is any weak solution of the Navier-Stokes equation. Such results will give no ultimate solution to the problem, but they are hoped to be useful for further investigation of the problem.

## 2. Statement of Theorems

In what follows $\Omega$ is a bounded domain in $R^m$ with smooth boundary $\partial\Omega$. The Navier-Stokes equation for an incompressible fluid with density one may be written formally

$$\text{(NS)}\quad \partial_t u - \nu\Delta u + (u\cdot\text{grad})u + \text{grad } p = f,$$
$$\text{div } u = 0,\ u_{|\partial\Omega} = 0,$$

where $u = u(t,x)$ is the velocity field, p the pressure, f the external force, $\nu > 0$ the (kinematic) viscosity, and $\partial_t = \partial/\partial t$. We assume ($f = f_\nu$ may depend on $\nu$)

$$\text{(2.1)}\quad f \in L^1((0,T);L^2(\Omega)) \text{ for any } T > 0.$$

Here and in what follows $L^2(\Omega)$ may denote, indiscriminately, the $L^2$-space of scalar, vector, or tensor-valued functions and similarly for other function spaces such as $C^k(\Omega)$, $H^s(\Omega)$ (Sobolev spaces).

A weak solution u to (NS) is assumed to satisfy the following conditions, where V is the space of vector-valued $H^1_0(\Omega)$-functions with divergence zero. (We write $\partial_k = \partial/\partial x_k$, and ( , ) [$\|\ \|$] denotes the (formal) inner product [norm] in $L^2(\Omega)$.)

$$\text{(2.2)}\quad u \in C_w([0,T];L^2(\Omega)) \cap L^2((0,T);V) \text{ for any } T > 0.$$

$$\text{(2.3)}\quad \|u(t)\|^2/2 + \nu\int_0^t \|\text{grad } u\|^2 dt \leqslant \|u(0)\|^2/2 + \int_0^t (f,u)dt.$$

$$\text{(2.4)}\quad (u(t),\varnothing(t)) - (u(0),\varnothing(0)) = \int_0^t [(uu,\text{grad } \varnothing) - \nu(\text{grad } u,\text{grad } \varnothing) + (f,\varnothing) + (u,\partial_t\varnothing)]\,dt$$

for every vector-valued test function $\varnothing \in C^1([0,\infty] \times \bar{\Omega})$ satisfying div $\varnothing = 0$ and vanishing on $\partial\Omega$.

Remark 2.1

(a) It is known (see Leray [6], Hopf [7], Ladyzenskaya [8], Lions [9], Temam [10]) that a weak solution u exists for any $u(0) \in L^2(\Omega)$ with div $u(0) = 0$. We assume that for each $\nu > 0$, one such weak solution $u = u_\nu$ of (NS) has been chosen. For simplicity the parameter $\nu$ and the space variable x are suppressed, as is the time variable t frequently (as in the integrand in (2.4)).

(b) $C_w$ in (2.2) indicates weak continuity.

(c) In (2.4) the following short-hand notation is used:

$$\text{(2.5)}\quad (uu,\text{grad } \varnothing) = \sum_{j,k}(u_j u_k,\partial_k\varnothing_j) = -\sum_k(u_k\partial_k u_j,\varnothing_j).$$

(d) The test function $\varnothing$ in (2.4) is sometimes (as in Hopf [7]) assumed to have spatial compact support. To admit more general $\varnothing$ stated above, we may use the fact that (spatial) test functions with compact supports are dense in $W_0^{1,p}(\Omega)$ with divergence zero, for any $p < \infty$, which can be proved by the "pulling-in" method given by Heywood [11]. Indeed, the functions uu, grad u, etc., appearing in (2.4) belong (for each fixed t) to some $L^q(\Omega)$ with $q > 1$.

(e) (2.3) describes the energy inequality. Note that we do not require that the energy inequality hold on intervals $[t_0,t]$ with $t_0 > 0$, a condition necessary in other problems related to (NS) such as Leray's structure theorem for turbulent solutions.

The Euler equation is obtained from (NS) by formally setting $\nu = 0$. In general ($m \geqslant 3$), only local (in time) solutions are known for the Euler equation. We denote by $\bar{u}$ such a solution:

$$\text{(E)}\quad \partial_t\bar{u} + (\bar{u}\cdot\text{grad})\bar{u} + \text{grad } \bar{p} = \bar{f},\ 0 \leqslant t \leqslant \bar{T} < \infty,$$
$$\text{div } \bar{u} = 0,\ \bar{u}_{n|\partial\Omega} = 0 \text{ (normal component)}.$$

Existence and uniqueness of a smooth local solution $\bar{u}$ have been proved by many authors (Ebin-Marsden [12], BourguignonBrezis [13], Temam [14], Lai [15], Kato [16], Kato-Lai [17], and others). Thus, we may assume

(2.6) $\bar{u}, \bar{p}, \bar{f} \in C^1([0,\bar{T}] \times \bar{\Omega})$.

We note that $\bar{T}$ may be taken arbitrarily large if $m = 2$ and $\bar{u}(0) \in C^{1+\varepsilon}(\bar{\Omega})$, div $\bar{u}(0) = 0$.

We are now able to state the main theorem.

*Theorem I*

*Fix* $T > 0$, $T \leq \bar{T}$, *and assume*

(2.7) $u(0) \to \bar{u}(0)$ *in* $L^2(\Omega)$ *as* $\nu \to 0$.

(2.8) $\int_0^T \|f - \bar{f}\| dt \to 0$ *as* $\nu \to 0$.

*Then the following conditions* (i) *to* (iv) *are equivalent. (All limiting relations refer to* $\nu \to 0$.)

(i) $u(t) \to \bar{u}(t)$ *in* $L^2(\Omega)$, *uniformly in* $t \in [0,T]$

(ii) $u(t) \to \bar{u}(t)$ *weakly in* $L^2(\Omega)$ *for each* $t \in [0,T]$.

(iii) $\nu \int_0^T \|\text{grad } u\|^2 dt \to 0$.

(iii') $\nu \int_0^T \|\text{grad } u\|^2_{\Gamma_{c\nu}} dt \to 0$,

*where* $\| \ \|_{\Gamma_{c\nu}}$ *denotes the* $L^2(\Gamma_{c\nu})$*-norm,* $\Gamma_{c\nu} \subset \Omega$ *being the boundary strip of width* $c\nu$, *with* $c > 0$ *fixed but arbitrary.*

*If in particular* $\bar{f} = 0$, *these conditions are equivalent to*

(iv) $u(T) \to \bar{u}(T)$ *weakly in* $L^2(\Omega)$.

*Theorem Ia*

*Replace* $L^1$ *by* $L^2$ *in* (2.1), *and replace* (2.8) *by*

(2.8a) $\int_0^{T'} \|f-\bar{f}\|^2 dt \to 0$ *for some* $T' > T$, $T' \leq \bar{T}$.

*Then the equivalent conditions* (i) *to* (iii) *in Theorem I are implied by*

(v) $\int_0^{T'} \|u - \bar{u}\|^2 dt \to 0$.

<u>Remark 2.2</u>

(a) Condition (iii) states that the energy dissipation during a finite time tends to zero as $\nu \to 0$, and (iii') states that the dissipation within a boundary strip of width $c\nu$ tends to zero.

(b) From the practical point of view, these conditions do not appear very helpful in deciding whether or not the convergence (i) takes place. In fact, we do not know whether (iii) or (iii') is always true, always false (except in trivial cases), or both possibilities exist. It may be noted, however, that if the convergence does <u>not</u> take place, the energy dissipation within the boundary layer of width $c\nu$ must remain finite as $\nu \to 0$. Since the boundary layer is believed to have thickness proportional to $(\nu t)^{1/2}$, this suggests that something violent must have happened for small $t > 0$.

(c) In Theorems I and Ia, the family $\{u\}$ with the continuous parameter $\nu$ may be replaced by a sequence $\{u^n\}$ corresponding to a sequence $\nu_n \to 0$ of the parameter $\nu = \nu_n$.

## 3. <u>Proof of Easier Parts of the Theorems</u>

First we deduce simple consequences of the properties of the solutions $u$ and $\bar{u}$. For simplicity we use the notation $||| \ |||_p$ for the $L^p((0,T);L^2(\Omega))$-norm, and $(( \ , \ ))$ for the (formal) scalar product in $L^2((0,T);L^2(\Omega))$. These are used indiscriminately for scalar, vector, and tensor-valued functions. $K$ denotes various constants independent of $\nu$.

It is well known (and is easy to prove) that (2.3), (2.7) and (2.8) imply

(3.1) $|||u|||_\infty \leq \|u(0)\| + |||f|||_1 \leq K$,

(3.2) $\nu \, |||\text{grad } u|||_2^2 \leq \|u(0)\|^2/2 + |||u|||_\infty |||f|||_1 \leq K$.

Similarly, (E) and (2.6) imply

(3.3) $(\bar{u}\bar{u}, \text{grad } \bar{u}) = 0,$

(3.4) $\|\bar{u}\|^2/2 = \|\bar{u}(0)\|^2/2 + \int_0^t (\bar{f},\bar{u})dt.$

Now we shall prove simple implications contained in the theorems.

(a) (i) implies (ii). This is trivial.

(b) (ii) implies (iii). If (ii) is true, (2.3) gives

(3.5) $\limsup \nu |||\text{grad } u|||_2^2$

$\leqslant \limsup[((f,u)) - (\|u(T)\|^2 - \|u(0)\|^2/2]$

$\leqslant ((\bar{f},\bar{u})) - (\|\bar{u}(T)\|^2 - \|\bar{u}(0)\|^2)/2 = 0$

by (2.7), (2.8), (3.1) and (3.4); note that $\liminf \|u(T)\| \geqslant \|\bar{u}(T)\|$ because $u(T) \to \bar{u}(T)$ weakly. To see that $((f,u)) \to ((\bar{f},\bar{u}))$, use dominated convergence in t.

(c) (iii) implies (iii') trivially.

(d) (iv) implies (iii) if $\bar{f} = 0$, since only $u(T) \to \bar{u}(T)$ weakly is needed in the above proof in (b). Indeed, $|||f|||_1 \to 0$ implies $((f,u)) \to 0$ by (3.1).

(e) (v) implies (iii) under the additional assumptions stated. To see this, we integrate (2.3) in $t \in (0,T')$ to obtain

(3.6) $\limsup \nu\int_0^{T'} (T'-t)\|\text{grad } u\|^2 dt$

$\leqslant \limsup [\int_0^{T'} (T'-t)(f,u)dt + (T'\|u(0)\|^2 - |||u|||_2^2)/2],$

where $|||\ |||_2$ is taken on $(0,T')$. If (2.8a) and (v) are true, then $\int(T'-t)[(f,u)-(\bar{f},\bar{u})]dt \to 0$ and $|||u|||_2 \to |||\bar{u}|||_2$ so that the right member of (3.6) does not exceed

$$\int_0^{T'} (T'-t)(\bar{f}-\bar{u})dt + (T'\|\bar{u}(0)\|^2 - |||\bar{u}|||_2^2)/2$$

$$= \int_0^{T'} [\int_0^t (\bar{f},\bar{u})dt + (\|u(0)\|^2 - \|u\|^2)/2]dt = 0$$

by (3.4). It follows that the left member of (3.6) is zero. Since $T' > T$, this implies (iii).

## 4. Boundary Layers

In the proof of the remaining assertion that (iii') implies (i), we need a "boundary layer" v, which is a correction term (depending on $\nu$) to be subtracted from $\bar{u}$ to satisfy the zero boundary condition and which has a thin support. It may be noted at this point that v has no direct relation with the true boundary layer belonging to u. In fact, the latter is virtually unknown in the mathematical sense.

To construct v, we first introduce a smooth "vector potential" $\bar{a}$, defined on $[0,T] \times \bar{\Omega}$, such that

(4.1) $\bar{u} = \text{div } \bar{a}$ on $\partial\Omega$, $\bar{a} = 0$ on $\partial\Omega$.

$\bar{a}$ is a skew-symmetric tensor of second rank, and div $\bar{a}$ is a vector with components $\sum_k \partial_k \bar{a}_{jk}$. The existence of such an $\bar{a}$ will be proved in Appendix.

We next introduce a smooth cut-off function $\zeta: \mathbb{R}^+ \to \mathbb{R}^+$ such that

(4.2) $\zeta(0) = 1, \ \zeta(r) = 0$ for $r \geqslant 1,$

and set

(4.3) $z = z(x) = \zeta(\rho/\delta)$, where $\rho = \text{dist}(x,\partial\Omega),$

with a small parameter $\delta > 0$, which is assumed to tend to zero with $\nu$ with a rate to be determined below.

The boundary layer v is defined by

$$v = \operatorname{div}(z\bar{a}) = z \operatorname{div} \bar{a} + \bar{a}\cdot\operatorname{grad} z, \tag{4.4}$$

where $\bar{a}\cdot\operatorname{grad} z$ is a vector with components $\sum_k \bar{a}_{jk}\partial_k z$. Thus, v has a thin support near $\partial\Omega$ and satisfies

$$v = \bar{u} \text{ on } \partial\Omega, \quad \operatorname{div} v = 0 \text{ in } \Omega. \tag{4.5}$$

(Note that div div b = 0 if b is skew-symmetric.)

The following estimates for v can easily be established.

$$\begin{aligned} &\|v\|_{L^\infty} \leqslant K, \ \|v\| \leqslant K\delta^{1/2}, \ \|\partial_t v\| \leqslant K\delta^{1/2}, \\ &\|\operatorname{grad} v\|_{L^\infty} \leqslant K\delta^{-1}, \ \|\operatorname{grad} v\| \leqslant K\delta^{-1/2}, \\ &\|\rho \operatorname{grad} v\|_{L^\infty} \leqslant K, \ \|\rho^2 \operatorname{grad} v\|_{L^\infty} \leqslant K\delta, \\ &\|\rho \operatorname{grad} v\| \leqslant K\delta^{1/2}. \end{aligned} \tag{4.6}$$

Indeed, these estimates are obviously true for v replaced with z, together with analogous estimates involving second derivatives of z. Then (4.6) follows easily because $\bar{a} = 0$ on $\partial\Omega$. $\bar{a}$ and $\partial_t\bar{a}$ are smooth and vanish on $\partial\Omega$.

5. <u>Proof of (iii') ⇒ (i)</u>

We now assume (iii') and estimate $\|u-\bar{u}\|^2$ using (2.3), (2.7) and (3.4):

$$\begin{aligned} \|u-\bar{u}\|^2 &= \|u\|^2 + \|\bar{u}\|^2 - 2(u,\bar{u}) \\ &\leqslant \|u(0)\|^2 + 2\int_0^t (f,u)dt + \|\bar{u}(0)\|^2 + 2\int_0^t (\bar{f},\bar{u})dt - 2(u,\bar{u}) \\ &\leqslant o(1) + 2\int_0^t [(f,u) + (\bar{f},\bar{u})]dt + 2\|\bar{u}(0)\|^2 - 2(u,\bar{u}-v), \end{aligned} \tag{5.1}$$

where v is defined in the previous section and where o(1) denotes a quantity that tends to zero as $\nu \to 0$ uniformly in $t \in [0,T]$. Note that $\|v\| \leqslant K\delta^{1/2}$ by (4.6) and $\delta \to 0$ with $\nu$, and that $\|u\|$ $\leqslant K$ by (3.1). We have introduced the boundary layer v into the last term of (5.1) in order to facilitate the following estimates.

To estimate the last term on the right of (5.1), we use $\varnothing = \bar{u} - v$ as a test function in (2.4); this is allowed since $\bar{u} - v$ is smooth with $\operatorname{div}(\bar{u}-v) = 0$ and vanishes on $\partial\Omega$. The result is, when multiplied with $-2$,

$$\begin{aligned} -2(u,\bar{u}-v) + 2\|u(0)\|^2 = o(1) + \int_0^t &[-2(uu,\operatorname{grad}(\bar{u}-v) \\ &+ 2\nu (\operatorname{grad} u,\operatorname{grad}(\bar{u}-v)) - 2(f,\bar{u}) - 2(u,\partial_t(\bar{u}-v))]dt; \end{aligned} \tag{5.2}$$

note that $\|u(0)-\bar{u}(0)\| \to 0$ and $\|v(t)\| \to 0$.

The last term in the integrand in (5.2) is estimated as

$$\begin{aligned} -2(u,\partial_t(\bar{u}-v)) &= -2(u,\partial_t\bar{u}) + o(1) \\ &= o(1) + 2(u,(\bar{u}\cdot\operatorname{grad})\bar{u}) - 2(u,\bar{f}) \end{aligned} \tag{5.3}$$

(see (3.1), (4.6) and (E).) It follows from (5.1) to (5.3) that

$$\begin{aligned} \|u-\bar{u}\|^2 &\leqslant o(1) + 2\int_0^t [(f-\bar{f},u-\bar{u}) - (uu,\operatorname{grad}(\bar{u}-v)) \\ &\quad + (u,(\bar{u}\cdot\operatorname{grad})\bar{u}) + \nu (\operatorname{grad} u,\operatorname{grad}(\bar{u}-v))]\ dt \\ &\leqslant o(1) + 2\int_0^t [(f-\bar{f},u-\bar{u}) - ((u-\bar{u})(u-\bar{u}),\operatorname{grad} \bar{u}) \\ &\quad + (uu,\operatorname{grad} v) + \nu (\operatorname{grad} u,\operatorname{grad}(\bar{u}-v)]\ dt, \end{aligned} \tag{5.4}$$

where we have used the equality

$$(uu,\operatorname{grad} \bar{u}) - (u,(\bar{u}\cdot\operatorname{grad})\bar{u}) = ((u-\bar{u})(u-\bar{u}),\operatorname{grad} \bar{u}), \tag{5.5}$$

which follows easily from $\operatorname{div} u = \operatorname{div} \bar{u} = 0$, $u \in H_0^1(\Omega)$, $\bar{u}_{n\,|\,\partial\Omega} = 0$.

In view of the simple inequalities

$$(f-\bar{f},u-\bar{u}) \leqslant \|f-\bar{f}\| \ \|u-\bar{u}\| \leqslant K\|f-\bar{f}\|,$$
$$((u-\bar{u})(u-\bar{u}),\text{grad } \bar{u}) \leqslant K\|u-\bar{u}\|^2,$$

we obtain from (5.4) the following integral inequality:

$$(5.6) \qquad \|u-\bar{u}\|^2 \leqslant o(1) + \int_0^t [K\|u-\bar{u}\|^2 + R(t)]\,dt,$$

where

$$(5.7) \qquad R(t) = (uu,\text{grad } v) + \nu\,(\text{grad } u,\text{grad}(\bar{u}-v)) + K\|f-\bar{f}\|.$$

The integral inequality (5.6) for $\|u-\bar{u}\|^2$ is of a familiar type. It will lead to the desired result $\|u-\bar{u}\|^2 = o(1)$ if we can show that

$$(5.8) \qquad \int_0^t R(t)\,dt \leqslant o(1).$$

To prove (5.8), we first note that

$$|(uu,\text{grad } v)| \leqslant \|\rho^{-1}u\|^2_{\Gamma_\delta} \|\rho^2\text{grad } v\|_{L^\infty} \leqslant K\delta\|\text{grad } u\|^2_{\Gamma_\delta}$$

by (4.6) and the well-known inequality of Hardy-Littlewood (note that $u \in H^1_0(\Omega)$); we can take $\|\rho^{-1}u\|_{\Gamma_\delta}$ only on the boundary strip $\Gamma_\delta$ because v is supported on $\Gamma_\delta$. Similarly,

$$|\nu(\text{grad } u,\text{grad}(\bar{u}-v))|$$
$$\leqslant \nu\,\|\text{grad } u\|\ \|\text{grad } \bar{u}\| + \nu\,\|\text{grad } u\|_{\Gamma_\delta}\ \|\text{grad } v\|_{\Gamma_\delta}$$
$$\leqslant K\nu\,\|\text{grad } u\| + K\nu\delta^{-1/2}\,\|\text{grad } u\|_{\Gamma_\delta}$$

by (4.6).

If we simply set $\delta = c\nu$, we thus obtain

$$R(t) \leqslant K\nu\,\|\text{grad } u\|^2_{\Gamma_{c\nu}} + K\nu\,\|\text{grad } u\| + K\nu^{1/2}\,\|\text{grad } u\|_{\Gamma_{c\nu}}$$
$$+ K\,\|f-\bar{f}\|.$$

From this (5.8) follows by (iii') and (2.8), since

$$\int_0^t \nu\,\|\text{grad } u\|dt \leqslant t^{1/2}\,\nu\,|||\text{grad } u||| = O(\nu^{1/2})$$

by (3.2).

## Appendix
## Construction of the Vector Potential

*Lemma* A1

*Let* u *be a smooth tangential vector field on a smooth closed surface* $\Gamma$ *in* $R^m$. *There exists a skew-symmetric tensor field* a *of second rank on* $R^m$ *such that* a = 0 *and* div a = u *on* $\Gamma$ ($\partial_k a_{jk} = u_j$ *in tensor notation*). *If* u *depends on a parameter* t *smoothly,* a *can be chosen similarly.*

Proof

If $\Gamma$ is the plane $x_1 = 0$, $u = u(x')$ is defined for $x' = (x_2,\dots,x_m)$ with $u_1 = 0$. If we set $a_{j1} = -a_{1j} = x_1 u_j(x')$, $a_{11} = 0$, and $a_{jk} = 0$ for j, k $\geqslant$ 2, a satisfies the required conditions.

In the general case, the problem is locally reduced to the special case just considered by a coordinate transformation with Jacobian determinant 1. Then we may identify u and a with an (m-1) form and an (m-2) form, respectively, so that div a = u has an invariant meaning da = u. Thus, we can construct an a with the required properties in a neighborhood of each point of $\Gamma$.

Next we construct a in a neighborhood of $\Gamma$. To this end we use a partition of unity $\{\varnothing^s\}$ in a neighborhood of $\Gamma$ such that on the support of each $\varnothing^s$, a local solution $a^s$ can be constructed as above. Setting $a = \sum_s \varnothing^s a^s$ then gives the desired a. Indeed, it is obvious that $a = 0$ on $\Gamma$, and $\operatorname{div} a = \sum_s [\varnothing^s \operatorname{div} a^s + (\operatorname{grad} \varnothing^s) a^s]$ $= \sum_s \varnothing^s u = u$ on $\Gamma$.

Finally, we extend $a = a_0$ thus obtained to all of $R^m$. It suffices to introduce a smooth cut-off function $\zeta$ and set $a = \zeta a_0$. Here $\zeta$ should be equal to 1 in a neighborhood of $\Gamma$ and have support contained in the domain of $a_0$.

*Corollary*

u *can be extended to a vector field on* $R^m$ *with* div u = 0 *and* $u_{n|\Gamma} = 0$.

<u>Proof</u>

It suffices to set u = div a (note that div div a = 0).

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# Free Boundary Problems in Mechanics*


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## Abstract


Free boundary problems are defined and illustrated by several problems in mechanics. First the problem of finding the free surface of a liquid in hydrostatic equilibrium is considered. Then the effect of surface tension is taken into account. Finally, the contact of an inflated membrane, such as a balloon or tire, with a solid surface is formulated. This problem is solved by the method of matched asymptotic expansions when the contact area is small.



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