(E) Lagrangian Theory of Energy Dissipation for Navier-Stokes Turbulence?

In this section we consider - briefly - how these results might carry over to energy dissipation for incompressible fluid turbulence governed by the Navier-Stokes equation as $Re \to \infty$.

The phenomenon of spontaneous stochasticity must be presumed to carry over to the Navier-Stokes solutions in the limit as $\nu \to 0$ or $Re \to \infty$. In fact, we have seen that this phenomenon is just a manifestation of Richardson diffusion and the fact that initial particle separation Δ_0 is "forgotten" at long times. As discussed, this property can be observed rather convincingly in simulation and will, doubtless, be seen someday in laboratory experiment as well.

However, spontaneous stochasticity raises a number of serious puzzles for the traditional Lagrangian picture of 3D energy cascade, based on G. I. Taylor's idea of vortex line-stretching. Can there be material lies at all, if Lagrangian particle trajectories are stochastic? If the notion of a material loop still makes sense, then does the Kelvin Theorem hold in any approximate sense for the $Re \to \infty$ limit? These questions are still entirely unanswered. However, let us offer some reasonable conjectures. For more details, see

G. L. Eyink, "Turbulent cascade of circulations," C. R. Physique **7** 449-455 (2006)

G. L. Eyink, "Cascade of circulations in fluid turbulence," Phys. Rev. E. **74** 066302 (2006)

G. L. Eyink, "Turbulent diffusion of lines and circulations," Phys. Lett. A **368** 486-490 (2007)

In the first place we believe that there should also be a turbulent diffusion process of material loops. That is, there should be a nontrivial probability distribution $P_{\mathbf{u}}(C, t|C_0, t_0)$ for a loop C_0 at time t_0 to flow into a loop C at the later (or earlier) time t. The picture is:

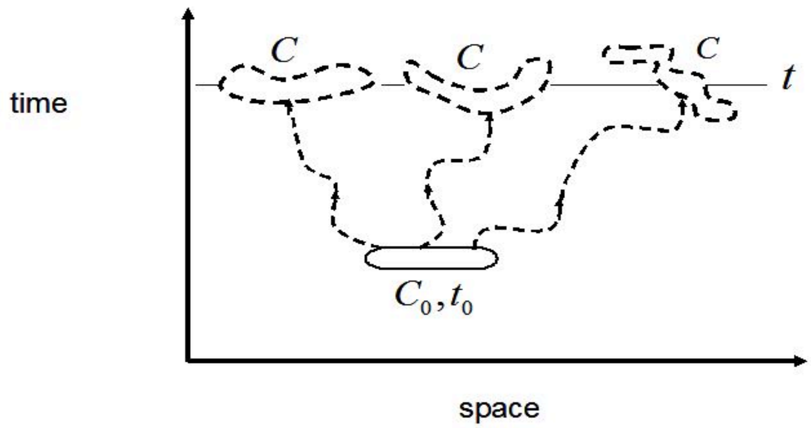

Figure 4.

This clearly requires a careful limiting procedure (e.g. smoothing the velocity, $\mathbf{u} \to \bar{\mathbf{u}}_\ell$, and then taking the limit $\ell \to 0$ after evolving $t_0 \to t$). Otherwise, taking $\ell \to 0$ first before $t_0 \to t$, one may expect the loop to "explode" into a disconnect cloud of particles!

This is one possible answer to the problem of the existence of material loops: they exist but became stochastic. What about the Kelvin Theorem? We offer the following conjecture, that circulations are conserved on average taking the expectation over the ensemble of material loops:

$$\Gamma(C,t) = \int P_{\mathbf{u}}(dC_0, t_0|C,t)\Gamma(C_0, t_)),\ t > t_0$$

Notice that this is the exact formal analog of equation (**) for the passive scalar. See the papers of Eyink (2006, 2007) for more details. Furthermore, this conjecture follows if the stochastic Kelvin Theorem proved by Constantin-Iyer for $\nu > 0$ holds even in the limit $\nu \to 0$.

If this conjecture is correct, it still remains to connect this "generalized Kelvin Theorem" to energy cascade in a precise way. We believe however that this is a key problem in theoretical turbulence of the 21st century. It is interesting in this context to quote two famous 20th century scientists, who struggled hard with these issues. The first is G. I. Taylor himself who , in a 1975 interview, discussed some of his early work on turbulence:

> "Though in 1937 I had realized the equivalence of the correlation description of turbulence and the spectrum description, my idea of the dynamics was directed to trying to connect the rate of increase of mean-square vorticity with dispersion, because if two neighboring points on a vortex line are separating the vorticity is increasing, and of course the rate of dissipation of energy is increasing."

But Taylor concludes:

> "However, I did not see how to turn this idea into a mathematical description which could form the basis of a theory and could predict things that could be verified or disproved experimentally."
>
> - from G. K. Batchelor, "An unfinished dialogue with G. I. Taylor," J. Fluid Mech. **70** 625-638 (1975)

Nevertheless, Taylor's ideas have proved extremely influential and are now generally believed to be true, despite the many difficulties surrounding them. Let us quote from the famous undergraduate textbook of R. P. Feynman, who, in section 41-5 "The limit of zero viscosity" discusses the infinite Reynolds-number limit of turbulent flow:

> "You may be wondering, 'What is the fine-grain turbulence and how does it maintain itself? How can the vorticity which is made somewhere at the edge of the cylinder generate so much noise in the background?' The answer is again interesting. Vorticity has a tendency to amplify itself. If we forget for a moment about the diffusion of vorticity which causes a loss, the laws of flow say (as we have seen) that the vortex lies are carried along with the fluid, at the velocity $\mathbf{v}$. We can imagine a certain number of lines of $\mathbf{\Omega}$ which are being distorted and twisted by the complicated flow pattern of $\mathbf{v}$. Thus pulls the lines closer together and mixes them all up. Lines that were simple before will get knotted and pulled close together. They will

be longer and tighter together. The strength of the vorticity will increase and its irregularities - the pluses and minuses - will, in general, increase. So the magnitude of vorticity in three dimensions increases as we twist the fluid about."

- Richard Feynman, quoted from R. P. Feynman R. B. Leighton, and M. Sands The Feynman Lectures on Physics (Addison-Wesley Publishing Co., Reading, MA, 1964). Volume II, Section 41-5

These views seem very plausible and must contain the germ of the truth. However, we have seen that, upon reflection, the matter is very subtle and difficult. It is not even clear that a simple, natural idea - that "the strength of the vorticity will increase ...[as vortex lines get knotted and pulled together]" - is true, at least in the most naive sense. However, experiments and simulations are getting better. Also, our mathematical tools have been considerably sharpened. Perhaps the 21st century shall be the one in which all those questions are, finally, answered!!!