(D) Scalar Cascade and Spontaneous Stochasticity

See T&L, Section 8.6

A passive scalar contaminant in a fluid, governed by the scalar advection-diffusion equation

$$\partial_t \theta(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x \theta(\mathbf{x},t) = \kappa \triangle \theta(\mathbf{x},t)$$

has an infinite number of conservation laws in the ideal limit $\kappa \to 0$, since, for any function $h:\mathbb{R}\to\mathbb{R}$

$$\partial_t h(\theta) + \boldsymbol{\nabla}\cdot[\mathbf{u}h(\theta) - \kappa \boldsymbol{\nabla} h(\theta)] = -\kappa h''(\theta)|\boldsymbol{\nabla}\theta|^2.$$

Thus, all the integrals $H(t) = \int d^dx\ h(\theta(\mathbf{x},t))$ are (formally) conserved as $\kappa \to 0$. The most basic of these is the scalar intensity or scalar energy with $h(\theta) = \frac{1}{2}\theta^2$ or

$$E_\theta(t) = \tfrac{1}{2}\int d^dx\ \theta^2(\mathbf{x},t).$$

This quantity for the case that $\theta = T - \bar{T}$, the temperature fluctuation field, was motivated by A. M. Obukhov (1949) who noted that, when multiplied by $-\rho c_P/\bar{T}^2$, it represents the entropy perturbation for a fluid with an ideal gas equation of state. It has a turbulent cascade dynamics of a type very similar to that of the kinetic energy of the velocity field. Under spatial filtering the scalar equation becomes

$$\partial_t \bar{\theta}_\ell + \boldsymbol{\nabla}_x\cdot[\bar{\mathbf{u}}_\ell\bar{\theta}_\ell + \mathbf{J}_\ell] = \kappa\triangle\bar{\theta}_\ell$$

with

$$\mathbf{J}_\ell = \overline{(\mathbf{u}\theta)}_\ell - \bar{\mathbf{u}}_\ell\bar{\theta}_\ell = \text{spatial transport of scalar by subscale advection}$$

From this coarse-grained effective equation, one obtains a balance equation for the scalar intensity in large-scales:

$$\partial_t(\tfrac{1}{2}\bar{\theta}_\ell^2) + \boldsymbol{\nabla}\cdot[\tfrac{1}{2}\bar{\theta}_\ell^2\bar{\mathbf{u}}_\ell + \bar{\theta}_\ell\bar{\mathbf{J}}_\ell - \kappa\boldsymbol{\nabla}(\tfrac{1}{2}\bar{\theta}_\ell^2)] = -\Pi_\ell^\theta - \kappa|\boldsymbol{\nabla}\bar{\theta}_\ell|^2$$

with

$$\Pi_\ell^\theta = -\boldsymbol{\nabla}\bar{\theta}_\ell\cdot\mathbf{J}_\ell,$$

the scalar energy flux to small-scales. We see that the scalar fluctuations are transferred to the unresolved scales when

$$\mathbf{J}_\ell \propto -\boldsymbol{\nabla}\bar{\theta}_\ell,$$

on average, that is, when the subscale transport is <u>down-gradient</u>. That is, the turbulent cascade of the scalar is <u>forward</u> to small scales when the space-transport is <u>diffusive</u> and tends to spread out and homogenize high concentrations of the scalar. As we shall see later, this Eulerian view of the cascade has an exact Lagrangian counterpart.

It is quite easy to derive estimates

$$\boldsymbol{\nabla}\bar{\theta}_\ell = O(\tfrac{\delta\theta(\ell)}{\ell}), \quad J_\ell = O(\delta u(\ell)\delta\theta(\ell))$$

in terms of scalar increments $\delta\theta(\ell;\mathbf{x}) = \sup_{|\mathbf{r}|<\ell} |\delta\theta(\mathbf{r};\mathbf{x})|$, by means of techniques that are now quite familiar. Thus, one finds that

$$\Pi_\ell^\theta = O(\tfrac{\delta u(\ell)\delta\theta^2(\ell)}{\ell}).$$

In terms of Hölder exponents h_u, h_θ of the velocity and scalar, respectively, one find that — pointwise —

$$h_u + 2h_\theta \leq 1$$

is necessary for non-vanishing scalar flux. In terms of space-average flux and pth-order scaling exponents $\zeta_p^u, \zeta_p^\theta$,

$$\sigma_p^u + \sigma_q^\theta + \sigma_r^\theta \leq 1$$

for any $p, q, r \geq 1$, satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, with $\sigma_p^u = \zeta_p^u/p$ and $\sigma_p^\theta = \zeta_p^\theta/p$. For example, with $p = q = r = 3$, $\sigma_3^u + 2\sigma_3^\theta \leq 1$.

The nonlinear transfer dominates only at sufficiently large scales where the scalar diffusivity κ can be ignored. Balancing

$$\Pi_\ell^\theta = O(\tfrac{\delta u(\ell)\delta\theta^2(\ell)}{\ell}), \quad \kappa|\boldsymbol{\nabla}\bar{\theta}_\ell|^2 = O(\kappa\tfrac{\delta\theta^2(\ell)}{\ell^2})$$

one can see that the crossover scale η_θ is determined as the length-scale ℓ such that

$$\ell\delta u(\ell) \cong \kappa.$$

At a point where the velocity has Hölder exponent h_u, this implies that

$$\eta_h^\theta \cong L(Pe)^{\frac{-1}{1+h_u}}$$

where

$$Pe = \tfrac{u_{rms}L}{\kappa}$$

is the so-called Péclet number. It is interesting that this length-scale depends only upon the velocity scaling and not that of the scalar itself!

There are a number of possible scalar cascade regimes, depending upon the relative sizes of $\eta_h = LRe^{-1/(1+h_u)}$ and $\eta_h^\theta = LPe^{-1/(1+h_u)}$, or, equivalently, the value of the Prandtl number

$$Pr = \frac{\nu}{\kappa}$$

[so-called when κ is thermal diffusivity, but called Schmidt number and denoted by Sc when κ is the mass diffusivity of a solute or airborne tracer]. There are conventional names for the ranges:

$$\underline{\text{inertial-convective:}}\ \ L \gg \ell \gg \eta_h, \eta_h^\theta\ (Re \gg 1, Pe \gg 1)$$

$$\underline{\text{viscous-convective:}}\ \ \eta_h \gg \ell \gg \eta_h^\theta\ (Pe \gg 1, Pr \gg 1)$$

$$\underline{\text{inertial-diffusive:}}\ \ \eta_h^\theta \gg \ell \gg \eta_h\ (Re \gg 1, Pr \ll 1)$$

$$\underline{\text{viscous-diffusive:}}\ \ \eta_h, \eta_h^\theta \gg \ell$$

For more information on these, see T&L, Section 8.6. Here we shall focus mainly on the "inertial-convective range" or "Obukhov-Corrsin range". In this case, the velocity field is in its usual high-Reynolds-number turbulent state. For example, within K41 description, $h_u = 1/3$. Using the scaling relation $\Pi_\ell^\theta \sim \delta u(\ell)\delta\theta^2(\ell)/\ell$, one sees that the condition for constant flux of the scalar energy is

$$\delta\theta(\ell) \sim \theta_{rms}(\ell/L)^{1/3}$$

with also $h_\theta = 1/3$. This scaling was first proposed by

A. M. Oboukhov, "Structure of the temperature field in turbulent flows," Izv. Akad. Nauk. SSSR, Geogr. and Geophys. **13** 58 (1949)

S. Corrsin, "On the spectrum of isotropic temperature fluctuations in isotropic turbulence," J. Appl. Phys. **22** 469 (1951)

In reality there is intermittency in the scalar cascade, leading to anomalous scaling

$$\langle \delta\theta^p(\mathbf{r})\rangle \sim \theta_{rms}^p(\tfrac{r}{L})^{\zeta_p^\theta}.$$

In fact, it appears from experiment and simulation that intermittency is more severe for the scalar than for the (longitudinal) velocity! For some recent simulation data, see

T. Watanabe &T. Gotoh, "Statistics of a passive scalar in homogeneous turbulence," New J. Phys. **6** 4 (2004)

We reproduce Figure 29 from that paper:

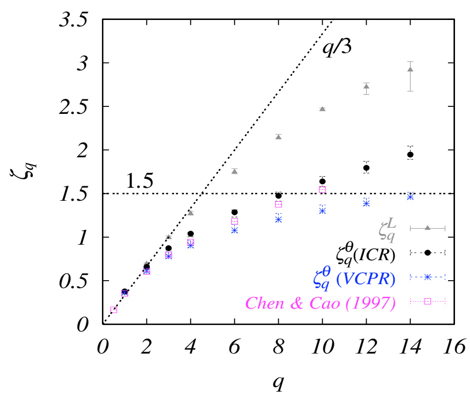

Figure 29. Comparison of the scaling exponents for run 2. ICR is for the inertial-convective range $200 < r/\eta < 400$, and VCPR is for the viscous-convective precursor range $30 < r/\eta < 60$. C & C is the DNS data from Chen and Cao [45].

One of the exciting developments of the 1990's is that the scaling exponents ζ_p^θ were calculated analytically in a model of passive advection by a synthetic turbulent velocity field that was taken to be a Gaussian random field with zero mean and covariance, for $r \ll L$, of the form

$$\langle \delta u_i(\mathbf{r},t)\delta u_j(\mathbf{r},t')\rangle = DP_{ij}(\boldsymbol{\nabla})r^\xi\delta(t-t')$$

with $P_{ij}(\boldsymbol{\nabla}) = \delta_{ij} - \partial_i\partial_j\triangle^{-1}$ the projection onto solenoidal velocity field and with an exponent $0 < \zeta < 2$ for the spatial scaling of the velocity field. This is called the Kraichnan model or the rapid-change velocity ensemble, which was first introduced by

R. H. Kraichnan, "Small-scale structure of a scalar field convected by turbulence," Phys. Fluids **11** 945-953 (1968)

who pointed out the key fact that there is no "closure problem" for this model and that scalar corrections obey exact, closed equations. For an extensive review on the analysis of this model, by some of the principal researchers, see:

> G. Falkovich, K, Gawędzki & M. Vergassola, "Particles and fields in fluid turbulence," Rev. Mod. Phys. **73** 913-975 (2001).

This work goes beyond the scope of this course, but students are encouraged to study the above review! We shall later discuss some other important discoveries made within the Kraichnan model that appear to be relevant to real turbulence.

Scalar cascade leads to the same type of "dissipative anomaly" for the scalar energy as was discussed earlier for the kinetic energy. An alternative approach to the large-scale balance of scalar energy is to consider the smooth "point-split" regularization

$$e_\ell^{*\theta}(\mathbf{x},t) = \tfrac{1}{2}\theta(\mathbf{x},t)\bar{\theta}_\ell(\mathbf{x},t)$$

which is easily shown to satisfy

$$\begin{aligned}\partial_t(\frac{1}{2}\theta\bar{\theta}_\ell) + \boldsymbol{\nabla}\cdot[(\frac{1}{2}\theta\bar{\theta}_\ell)\mathbf{u} + \frac{1}{4}\overline{[\mathbf{u}(\theta^2)]}_\ell - \frac{1}{4}\mathbf{u}\overline{(\theta^2)}_\ell - \kappa\boldsymbol{\nabla}(\frac{1}{2}\theta\bar{\theta}_\ell)]\\ = -\frac{1}{4\ell}\int d^dr\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})|\delta\theta(\mathbf{r})|^2 - \kappa\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\bar{\theta}_\ell\end{aligned} \qquad (14)$$

If one takes $\kappa\to 0$ first and $\ell\to 0$ second, one derives in the same manner as for the velocity field that, in the distribution sense,

$$\partial_t(\tfrac{1}{2}\theta^2) + \boldsymbol{\nabla}\cdot(\tfrac{1}{2}\theta^2\mathbf{u}) = -D_\theta(\mathbf{u},\theta) \qquad (\star)$$

with

$$D_\theta(\mathbf{u},\theta) = \lim_{\ell\to 0}\tfrac{1}{4\ell}\int d^dr\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})|\delta\theta(\mathbf{r})|^2,$$

as long as $\mathbf{u},\theta\in L^3$ in spacetime. If one takes the opposite limit of $\ell\to 0$ first, at fixed $\kappa>0$, then one obtains

$$\partial_t(\tfrac{1}{2}\theta^2) + \boldsymbol{\nabla}\cdot[\tfrac{1}{2}\theta^2\mathbf{u} - \kappa\boldsymbol{\nabla}(\tfrac{1}{2}\theta^2)] = -\kappa|\boldsymbol{\nabla}\theta|^2.$$

If $\theta^\kappa\to\theta$ as $\kappa\to 0$ in the strong L^3 sense, then one gets $(\star)$ again, but now with

$$D_\theta(\mathbf{u},\theta) = \lim_{\kappa\to 0}\kappa|\boldsymbol{\nabla}\theta^\kappa|^2 \geq 0.$$

It is interesting that, in this form, $D_\theta(\mathbf{u},\theta)$ depends (explicitly) only on the scalar field.

Finally, it may also be inferred that

$$\lim_{r\to 0} \frac{\langle \delta u_L(\mathbf{r}) |\delta\theta(\mathbf{r})|^2\rangle_{ang}}{r} = -\frac{4}{d} D_\theta(\mathbf{u},\theta)$$

which is a spacetime local form of the scalar 4/3-law or Yaglom relation first derived by

A. M. Yaglom, "Local structure of the temperature field in a turbulent flow," Dokl. Akad. Nauk. SSSR **69** 743-746 (1949)

For a current test of this relation, globally in spacetime, we can refer to Figure 8 of Watanabe & Gotoh (2004):

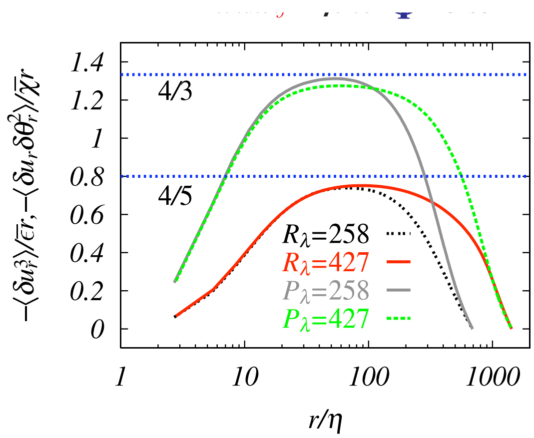

Figure 8. Approach of the curves to the $\frac{4}{5}$- and $\frac{4}{3}$-laws. Curves are plotted for $-\langle \delta u_r^3\rangle/(\bar{\epsilon} r)$ and $-\langle \delta\theta_r^2 \delta u_r\rangle/(\bar{\chi} r)$ for runs 1 and 2, respectively.

As one can see, Reynolds and Péclet numbers presently achievable are still not large enough to completely verify the relation.

The important conclusion of our discussion is that a passive scalar in a turbulent flow should also exhibit anomalous dissipation of scalar energy in the limit $\kappa \to 0$, or $Pe \to \infty$. Direct evidence for non-vanishing scalar dissipation in simulations and experiment is reviewed here:

Donzis, D., Sreenivasan, K. R., and Yeung, P. K. Scalar dissipation rate and dissipative anomaly in isotropic turbulence. J. Fluid Mech. **532**, 199–216 (2005)

In this study, the authors non-dimensionalized the mean scalar dissipation rate $\varepsilon_\theta = \langle \kappa |\nabla\theta|^2\rangle$ as

$$D_\theta = \frac{\varepsilon_\theta}{\theta_{rms}^2/T},$$

where $T = L^{2/3}/\langle\varepsilon\rangle$ is the large-eddy turnover time defined by the velocity integral length L and the mean kinetic-energy dissipation rate $\langle\varepsilon\rangle$. This scaling can be motivated by considering the Yaglom law at $r \simeq L$, in which case $\delta\theta(r) \simeq \theta_{rms}$ and $r/\delta u(r) \simeq T$. With this scaling, the dimensionless dissipation for different Schmidt numbers appears as:

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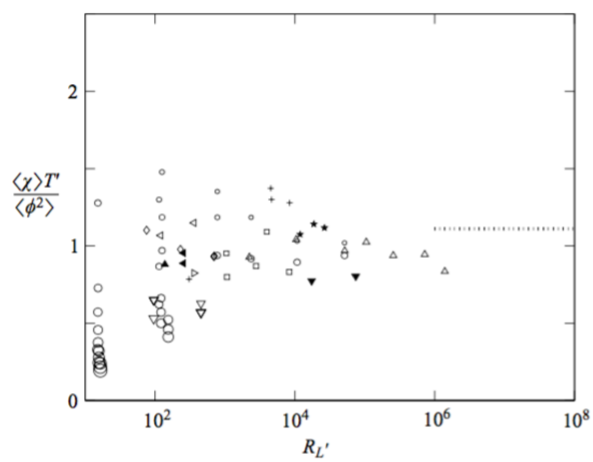


FIGURE 10. Scalar dissipation rate normalized by $T' = L^{2/3}/\langle\varepsilon\rangle^{1/3}$. Symbols as in figure 3. The relative size of the symbol illustrate the relative magnitude of Sc. Dotted line: the limit $2/3C_{OC}$ predicted by (3.19) with $C_{OC} = 0.6$.

The empirical results show not only an apparent asymptote for $Re \gg 1$, $Pe \gg 1$, but also decreasing dependence on Sc in the limit.

We have seen that the physical origin of the scalar dissipative anomaly— in the Eulerian description — is the diffusive mixing by the velocity field. As we shall now discuss, the scalar anomaly has been rigorously demonstrated to occur in the Kraichnan model. Furthermore, extremely interesting insights have been obtained there on the Lagrangian mechanism of the anomaly.

Spontaneous Stochasticity

We return to the consideration of Lagrangian particle evolution in a turbulent flow in the limit $\nu \to 0$, or $Re \to \infty$. In that limit, the velocity field becomes rough with Hölder exponent $0 < h < 1$, say, and a pair of particles can separate to a distance $\Delta^{(2)}(t)$ at time t bounded above by

$$\Delta^{(2)}(t) \leq [\Delta_0^{1-h} + (1-h)A(t-t_0)]^{\frac{1}{1-h}} \qquad (\star)$$

if the initial separation is Δ_0 at time t_0. We earlier considered the limit $t \to \infty$ and showed that, for $h < 1$, the initial particle separation Δ_0 is "forgotten" at long times. Another way to consider the same problem is to take the limit $\Delta_0 \to 0$ with time t fixed. In that case, one obtains

$$\Delta_0 \to 0 : \Delta^{(2)}(t) \leq [(1-h)A(t-t_0)]^{\frac{1}{1-h}}.$$

But, according to this estimate, it is possible that two particles started at the SAME point may separate to a finite distance at time $t > t_0$!!! This seems to contradict the expectation that there should be a unique solution of the equation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t), t), \;\; \mathbf{x}(t_0) = \mathbf{x}_0$$

However, uniqueness need not hold if $h < 1$. The inequality $(\star)$ does imply uniqueness if $h = 1$, since in that case

$$\Delta^{(2)}(t) \leq \Delta_0 e^{A(t-t_0)} \to 0, \;\text{ as } \Delta_0 \to 0$$

But for $h < 1$, clearly, no such conclusion may be drawn.

Simple examples, furthermore, show that such non-uniqueness really occurs. A standard example is, for $x \in \mathbb{R}_+$,

$$\frac{dx}{dt} = Ax^h, \;\; x(0) = 0$$

which has at least two distinct solutions:

$$\begin{aligned} x_1(t) &= [(1-h)At]^{\frac{1}{1-h}} \\ x_2(t) &\equiv 0. \end{aligned} \qquad (15)$$

As a matter of fact, this example has a continuum of solutions corresponding to a parameter

$\tau \geq 0$

$$x(t;\tau) = [(1-h)A(t-\tau)_+]^{\frac{1}{1-h}}$$

where $(s)_+ = s$ for $s \geq 0$ and $(s)_+ = 0$ for $s < 0$. The quantity τ corresponds to the duration of time that the particle "waits" at $x = 0$ before moving to the right. Thus, $x_1(t) = x(t;0)$ and $x_2(t) = x(t,\infty)$. For more on the phenomenon of non-uniqueness of solutions of ODE's with non-smooth (non-Lipschitz) velocities, see, for example,

P. Hartman, Ordinary Differential Equations (Wiley, New York, 1964), Chapter II.

We have come to a very important — and a bit distressing! — point in our deliberations. We know from Onsager's work that the turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ cannot remain Lipschitz, or even Hölder continuous with exponent $h > \frac{1}{3}$, in the limit $\nu \to 0$ or $Re \to \infty$. At least this is true if turbulent energy dissipation ε does not vanish in the limit, as experiments and simulations suggest. However, the most common explanation for the enhancement of energy dissipation in turbulent flow is G. I. Taylor's mechanism of vortex line-stretching. This mechanism assumes that vortex lines move like material lines and, more importantly, that circulations on material loops are conserved. But what is meant by a "material line" or a "material loop" if there is a non-uniqueness of Lagrangian trajectories? It is not even clear how to formulate the Kelvin Theorem in this case, let alone to determine whether it is true or false. At this point, there is no generally accepted answer these questions. We regard it as one of the most important outstanding problems in theoretical turbulence today.

Some very important advances on this problem have been made in the last decade, however, in the context of the Kraichnan white-noise advection model. Furthermore, it has been found there that non-uniqueness of Lagrangian trajectories is intimately related to the anomalous dissipation of passive scalar energy. We cannot give full details of these developments, which go beyond the scope of the course, but we briefly review here the main findings.

The most significant discovery in this context was the phenomenon of spontaneous stochasticity:

D. Bernard, K. Gawędzki & A. Kupiainen, "Slow modes in passive advection," J. Stat. Phys. **90** 519-569 (1998), Section 7; cond-mat/9706035

and also

K. Gawędzki, "Soluble models of turbulent advection," Lectures given at the workshop Random Media 2000, Madralin by Warsaw, June 19-26, 2000; `nlin.CD/0207058`

K. Gawędzki & M. Vergassola, "Phase transition in the passive scalar advection," Physica D **138** 63-90(2000); `cond-mat/9811399`

In the last two works the phenomenon was called "intrinsic stochasticity" instead. However, in a later work

M. Chaves et al., "Lagrangian dispersion in Gaussian self-similar velocity ensembles," J. Stat. Phys. **113** 643-692 (2003); `nlin/0303031`

the term "spontaneous stochasticity" has been suggested instead. We shall refer this latter terminology. The heuristic picture is as follows:

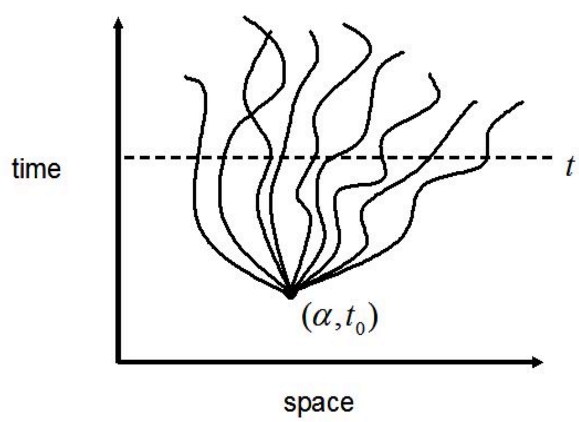


Figure 1

The term “spontaneous”stochasticity is used, since the equation ($\star$) is deterministic and involves no random element, for fixed $\mathbf{u}$. What is the origin of the randomness? One way to understand this is to obtain the probability distribution by regularizing the velocity field by filtering, so that particle trajectories of

$$\begin{aligned} \frac{d}{dt}\overline{\mathbf{x}}_\ell(t) &= \bar{\mathbf{u}}_\ell(\overline{\mathbf{x}}_\ell(t),t),\ t>t_0 \\ \overline{\mathbf{x}}_\ell(t_0) &= \boldsymbol{\alpha} \end{aligned} \tag{16}$$

are unique, but then to average over an ensemble of random initial conditions $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \Delta$ centered on $\boldsymbol{\alpha}$ and with variance ρ.

What is this “spontaneous stochasticity” that has been demonstrated for the Kraichnan model? The essential point is this: take a fixed velocity field realization $\mathbf{u}(\mathbf{x},t)$ from the Gaussian ensemble with exponent ξ in space and delta-correlated in time. These velocities are Hölder continuous with exponent $h=\xi/2$, $0<h<1$, with probability one. It is then found that the equation

$$\begin{cases} \frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t),t), t>t_0 \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \end{cases} \quad (*)$$

does not have a unique solution: there is a continuous infinity of Lagrangian trajectories. However, remarkably there is a unique random ensemble of solutions of ($*$). In particular, there is a nontrivial transition probability density

$$P_{\mathbf{u}}(\mathbf{x},t|\boldsymbol{\alpha},t_0) = \langle \delta^d(\mathbf{x}-\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}))\rangle_{\mathbf{u}}$$

where the average $\langle\cdot\rangle_{\mathbf{u}}$ is over the ensemble of solutions of ($*$). Thus, $P_{\mathbf{u}}(\mathbf{x},t|\boldsymbol{\alpha},t_0)$ gives the probability of observing the particle at $\mathbf{x}$ at time t which started at $\boldsymbol{\alpha}$ at time t_0. We emphasize that the velocity field $\mathbf{u}$ is fixed and there is no averaging over $\mathbf{u}$. This transition probability can be coarse-grained over initial particle locations with a kernel g_ρ :

$$P^{(\rho,\ell)}_{\mathbf{u}}(\mathbf{x},t|\boldsymbol{\alpha},t_0) = \int d\boldsymbol{\Delta}\ g_\rho(\boldsymbol{\Delta})\ \delta(\mathbf{x}-\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}+\boldsymbol{\Delta})).$$

This is the probability density for a particle to arrive to $\mathbf{x}$ at time t which was released at location $\boldsymbol{\alpha}+\boldsymbol{\Delta}$ at time 0 with density $g_\rho(\Delta)$.Thus, the picture becomes

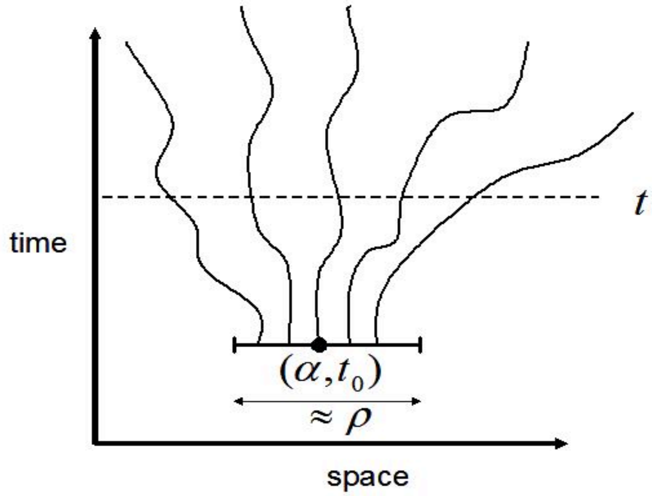


Figure 2.

It was been proved in the Kraichnan model that a limit exists if one takes first $\ell \to 0$ and then $\rho \to 0$

$$P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0) = \lim_{\rho\to 0} \lim_{\ell \to 0} P_{\mathbf{u}}^{(\rho,\ell)}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0)$$

Furthermore, the limit is independent of which filter kernel G_ℓ was used to regularize $\mathbf{u}$ and which distribution g_ρ was sampled to obtain initial conditions! These transition probability densities satisfy obvious conditions that $\int d^dx P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0) = 1$ and $P_{\mathbf{u}}(\mathbf{x}, t_0|\boldsymbol{\alpha}, t_0) = \delta^d(\mathbf{x}-\boldsymbol{\alpha})$, as well as the condition

$$\int d^d\alpha \; P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0) = 1$$

which expresses incompressibility of the fluid. Another important property is that (in a suitable sense)

$$(\partial_{t_0} + \mathbf{u}(\boldsymbol{\alpha}, t_0)\cdot \boldsymbol{\nabla}_\alpha)P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0) = 0$$

which can be seen formally by rewriting $P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0) = \langle \delta^d(\mathbf{x} - \mathbf{A}_t^{t_0}(\boldsymbol{\alpha}))\rangle_{\mathbf{u}}$ and using the equation $D_{t_0}\mathbf{A}_t^{t_0}(\boldsymbol{\alpha}) = 0$. This result is a consequence of the fact that $P_{\mathbf{u}}$ is supported on Lagrangian particle trajectories of the velocity field $\mathbf{u}$.

We have focused so far on forward evolution with $t > t_0$, but the equation $(*)$ can also be solved backward in time for $t < t_0$. The previous picture is then time-reversed:

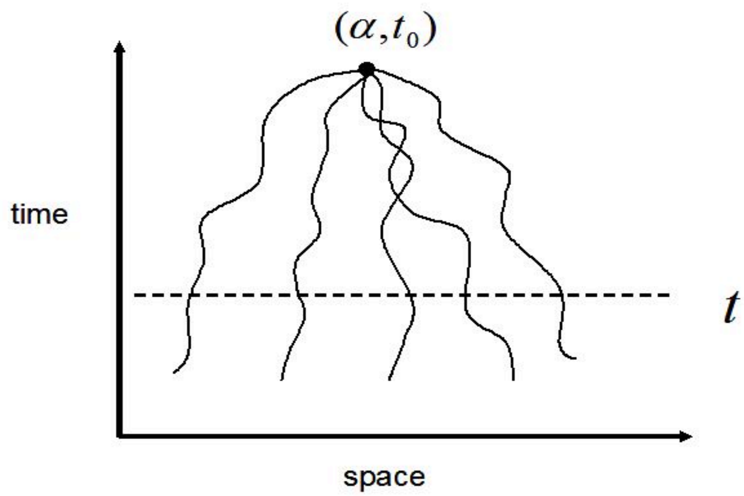


Figure 3.

The transition probabilities $P_{\mathbf{u}}(\mathbf{x}, t|\boldsymbol{\alpha}, t_0)$ make sense for any t, either $t > t_0$ or $t < t_0$. The properties that we have discussed above hold for both the future and for the past.

The above properties allow one to introduce solutions of the initial-value problem (IVP) of the scalar advection equation

$$\begin{cases} (\partial_t + \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\nabla}_x)\theta(\mathbf{x}, t) = 0, \quad t > t_0 \\ \theta(\mathbf{x}, t_0) = \theta_0(\mathbf{x}) \end{cases}$$

by averaging over Lagrangian particle trajectories backward in time

$$\theta(\mathbf{x}, t) = \int d^d\alpha\, \theta_0(\boldsymbol{\alpha}) P_{\mathbf{u}}(\boldsymbol{\alpha}, t_0|\mathbf{x}, t), \ t > t_0 \qquad (**)$$

It is remarkable that this set of solutions can be obtained by a number of physical limits. For example, if the velocity field is smoothed, $\mathbf{u} \to \bar{\mathbf{u}}_\ell$, then one can introduce $\theta^{(\ell)}$ as the solution of

$$\begin{cases} (\partial_t + \bar{\mathbf{u}}_\ell(\mathbf{x}, t) \cdot \boldsymbol{\nabla}_x)\theta^{(\ell)}(\mathbf{x}, t) = 0, \quad t > t_0 \\ \theta^{(\ell)}(\mathbf{x}, t_0) = \theta_0(\mathbf{x}) \end{cases}$$

and then it can be shown that

$$\theta(\mathbf{x}, t) = \lim_{\ell \to 0} \theta^{(\ell)}(\mathbf{x}, t).$$

Another important regularization is to add diffusion, yielding solutions θ^κ to the equation

$$\begin{cases} (\partial_t + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x)\theta^\kappa(\mathbf{x},t) &= \kappa\triangle\theta^\kappa(\mathbf{x},t) \\ \theta^\kappa(\mathbf{x},t_0) &= \theta_0(\mathbf{x}) \end{cases}$$

We recall that these can be represented by the formula

$$\theta^\kappa(\mathbf{x},t) = \int d^d\alpha\ \theta_0(\boldsymbol{\alpha})P^\kappa_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)$$

where $P^\kappa_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)$ is the transition probability for the stochastic ODE

$$\begin{cases} d\mathbf{X}^s_t(\mathbf{x}) &= \mathbf{u}^\nu(\mathbf{X}^s_t(\mathbf{x}),s)ds + \sqrt{2\kappa}d\mathbf{W}(s), \quad s<t \\ \mathbf{X}^t_t(\mathbf{x}) &= \mathbf{x} \end{cases}$$

solved backward in time. It can be shown that

$$P_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t) = \lim_{\nu,\kappa\to 0} P^{\nu,\kappa}_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)$$

and thus also that

$$\theta(\mathbf{x},t) = \lim_{\nu,\kappa\to 0}\theta^{\nu,\kappa}(\mathbf{x},t),$$

recovering the solution given in ($**$). In the Kraichnan model with an incompressible velocity field, it can be furthermore proved that limits are independent of the Prandtl number $Pr = \nu/\kappa$.

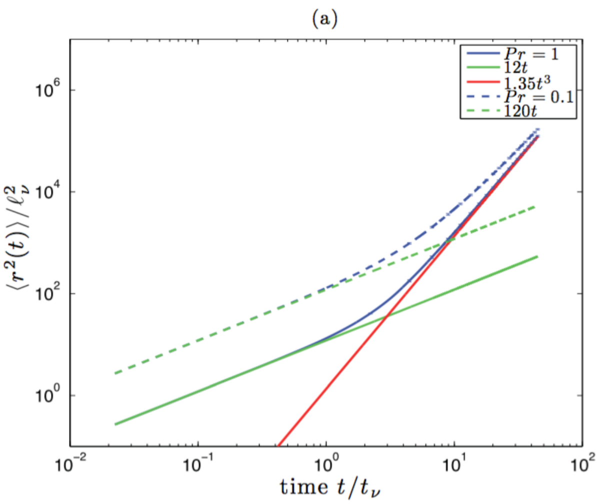
There is direct evidence of similar "spontaneous stochasticity" in incompressible fluid turbulence. We have already seen from the numerical results of Bitane et al. (2013) for Richardson dispersion that the mean-square distance $\langle[\Delta^{(2)}(t)]^2\rangle$ between particle pairs "forgets" the initial separation Δ_0 at long times. For stochastic Lagrangian particles subject to diffusivity κ, the comparable result was already observed in

G. L. Eyink, "Stochastic flux freezing and magnetic dynamo," Phys. Rev. E **83** 056405 (2011),

where stochastic Lagrangian particles were numerically calculated in the JHU database of homogeneous, isotropic turbulence:

`http://turbulence.pha.jhu.edu/Forced_isotropic_turbulence.aspx`

The mean-square separation backward in time was obtained at two different values of κ, $\kappa=\nu$ and $\kappa = 10\nu$:



As can be seen, the precise value of κ is "forgotten" after a time of order $(\kappa/\epsilon)^{1/2}$ and the mean-square dispersion at long times is in agreement with the Richardson t^3 prediction.

More recently, the paper

> T. D. Drivas and G. L. Eyink, "A Lagrangian fluctuation-dissipation relation for scalar turbulence. Part I. Flows with no bounding walls," J. Fluid Mech. **829** 153–189 (2017)

has made a similar study with the JHU database, but without averaging over initial points. The results are shown on the following page for two particular release points $\mathbf{x}$, the one shown on the right near a strong vortex and the other on the left in "background" turbulence. It can be seen that, even without averaging over release points, the mean-square dispersion exhibits an early-time diffusive growth $\propto 12\kappa t$ and then a super-ballistic growth at longer times. Also shown are the transition probability densities $p(y', 0|\mathbf{x}, t_f)$ for one of the particle Cartesian coordinates y'. Although one cannot take the limit $\nu \to 0$ using the database with a single value of Re, nevertheless $Re \gg 1$ and the transition probability densities are reasonably independent of $Pr = \nu/\kappa$, as predicted for incompressible fluid turbulence in the Kraichnan model.

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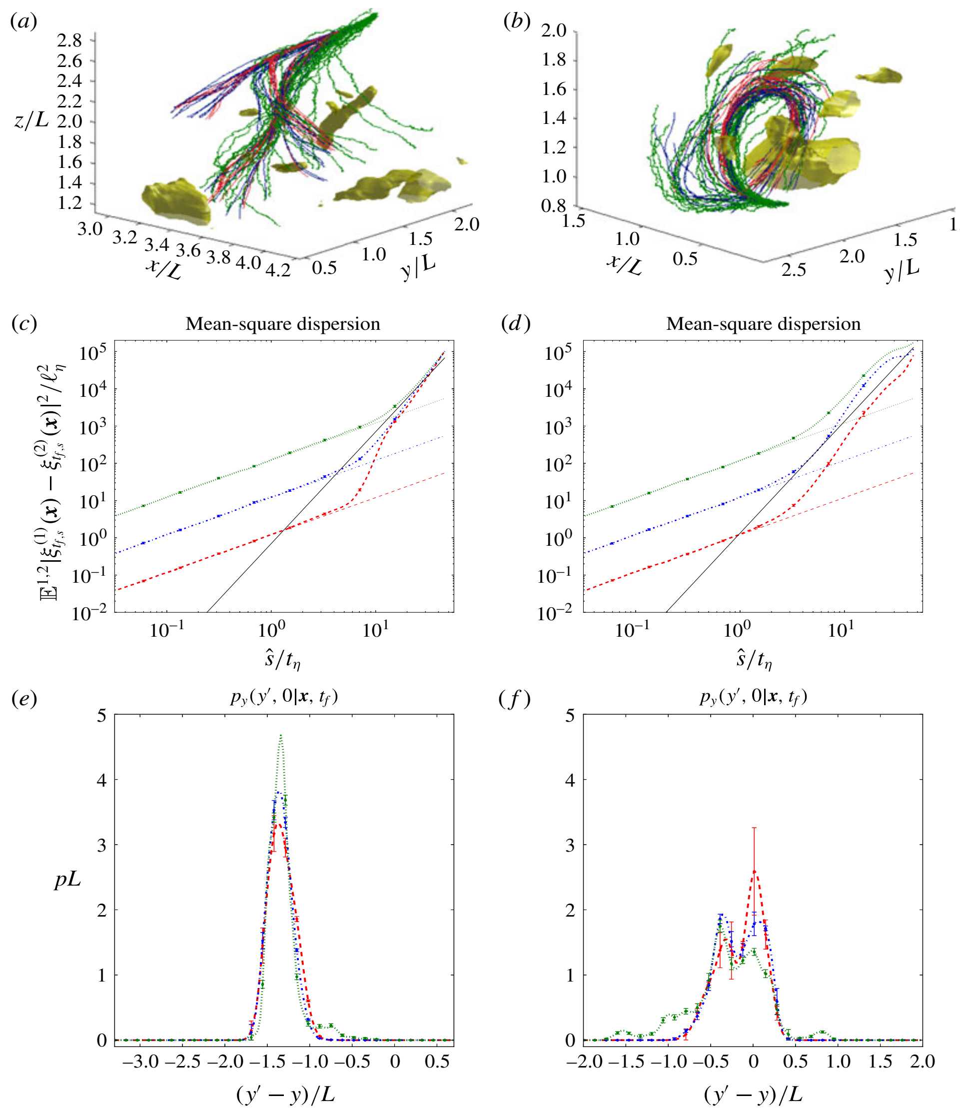

FIGURE 1. (Colour online) (a,c,e) Plots for release at $\boldsymbol{x} = (4.9637, 3.1416, 3.8488)$ in the background region; (b,d,f) plots for release at $\boldsymbol{x} = (0.2610, 3.1416, 1.4617)$ near a strong vortex. (a,b) Plots of 30 representative stochastic trajectories for $Pr = 0.1$ (green, light), 1.0 (blue, medium) and 10 (red, heavy) together with isosurfaces of coarse-grained vorticity $|\bar{\boldsymbol{\omega}}|T_L = 15$ at time $s = (2/3)T_L$. (c,d) Plots of particle dispersions (heavy) and short-time results $12\kappa\hat{s}$ (light) for each Pr, with $Pr = 0.1$ (green, dot, $\cdots\cdots$), 1.0 (blue, dash–dot, — · —) and 10 (red, dash, – – –), and a plot in (solid, ——) of $g\varepsilon\hat{s}^3$ with $g = 0.7$ (c) and $g = 4/3$ (d). (e,f) Plots of $p_y(y', 0|\boldsymbol{x}, t_f)$ for the three Pr values with the same line styles as in (c,d).

It might be expected that the scalar solutions given by $\theta(\mathbf{x},t) = \int d^d\alpha\ \theta_0(\boldsymbol{\alpha})P_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)$ shall be dissipative, when $P_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)$ is a non-deterministic distribution. This can be indeed shown directly. Take any convex function $h(\theta)$, e.g. $h(\theta) = \frac{1}{2}\theta^2$. Then, since "the average of the values is greater than the value at the average" (Jensen's inequality) for any convex function

$$h(\theta(\mathbf{x},t)) = h(\int d^d\alpha\ \theta_0(\boldsymbol{\alpha})P_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)) \leq \int d^d\alpha\ h(\theta_0(\boldsymbol{\alpha}))P_{\mathbf{u}}(\boldsymbol{\alpha},t_0|\mathbf{x},t)),\ \ t > t_0$$

Integrating over $\mathbf{x}$ and using the volume-preserving property then gives that

$$\int d^dx\ h(\theta(\mathbf{x},t)) \leq \int d^d\alpha\ h(\theta_0(\boldsymbol{\alpha})),\ \ t > t_0.$$

Hence, the integrals $H(t) = \int d^d\alpha\, h(\theta(\boldsymbol{\alpha},t))$ are decreasing in time, or dissipated! Thus, the formula $(**)$ provides the Lagrangian formulation of the scalar dissipation anomaly. The physics is very closely related to that seen in the Eulerian description, which is diffusive mixing. The formula $(**)$ shows that the solution $\theta(\mathbf{x},t)$ is an average over the values of the initial data $\theta_0(\mathbf{x})$, due to turbulent mixing produced by the stochastic Lagrangian trajectories.

For more discussions of these problems, see

W. E. and E. vanden-Eijnden, "Generalized flows, intrinsic stochasticity, and turbulent transport," Proc. Nat. Acad. Sci. **97** 8200-8205 (2000)

W. E. and E. vanden-Eijnden,"Turbulent Prandtl number effect on passive scalar advection," Physica D **152-153** 636-645(2001)

W. E. and E. vanden-Eijnden,"A note on generalized flows," Physica D **183** 159-174 (2003)

and, at a rigorous mathematical level,

Y. LeJan & O. Raimond, "Solutions statistiques fortes des équations différentielles stochastiques," C. R. Acad. Sci. Paris Sér. I. Math. **327** 893-896 (1998)

Y. LeJan & O. Raimond, "Integration of Brownian vector fields," Ann. Probab. **30** 826-873(2002)

Y. LeJan & O. Raimond, "Flows, coalescence and noise," Ann. Probab. **32** 1247-1315 (2004).

These papers give rigorous proofs for the Kraichnan model of the various results discussed above. In fact, as emphasized in the paper of Eyink & Drivas (2017) cited above, the original argument of Bernard et al. (1998) rigorously shows that spontaneous stochasticity is required for anomalous dissipation of a freely-decaying scalar, for any advecting velocity field whatsoever, and whether the scalar is active or passive. The key mathematical point here is that Jensen's inequality for a strictly convex function can become an equality if and only if the particle distribution function is deterministic (i.e. a delta function).

The paper of Eyink & Drivas (2017) has also proved mathematically for any advecting velocity field whatsoever and with possible time-dependent sources injecting the tracer, that anomalous scalar dissipation can occur if and only if Lagrangian particle trajectories become spontaneously stochastic. Even if the scalar is active (i.e. the velocity dynamics depends upon θ somehow), then spontaneous stochasticity is still necessary for anomalous dissipation. The proofs exploit an exact "Lagrangian fluctuation-dissipation relation" that was previously derived by

> Celani, A., Cencini, M., Mazzino, A. & Vergassola, M. 2004 Active and passive fields face to face. New J. Phys. **6** 72 (2004).

These statements about necessity of spontaneous stochasticity only hold away from solid walls, as discussed here:

> T. D. Drivas and G. L. Eyink, "A Lagrangian fluctuation-dissipation relation for scalar turbulence. Part I. wall-bounded flows," J. Fluid Mech. **829** 236–279 (2017)

In the presence of walls, thin scalar boundary layers are another possible mechanism of anomalous scalar dissipation. However, it can still be shown even in the presence of walls that spontaneous stochasticity is sufficient to produce anomalous for a passive scalar.