(C) Inertial-Range Lagrangian Dynamics & Lagrangian Intermittency

We have earlier discussed the Lagrangian dynamics associated to the coarse-grained/large-scale velocity field $\bar{\mathbf{u}}_\ell$, defined via the flow maps $\overline{\mathbf{X}}^t_{\ell,t_0}$ that satisfy

$$\begin{cases} \frac{d}{dt}\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}) &= \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}),t) \\ \overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}) &= \boldsymbol{\alpha} \end{cases}$$

For example, these appeared (implicitly) in our discussion of the inertial-range validity of the Kelvin Theorem, where the loop $\overline{C}_\ell(t)$ was defined as $\overline{\mathbf{X}}^t_{\ell,t_0}(C)$. The flows $\overline{\mathbf{X}}^t_{\ell,t_0}$ correspond to advection by all the turbulent eddies at length-scales $> \ell$. They satisfy all the usual properties of Lagrangian flow maps, such as the semi-group property, volume-preserving, etc. One may thereby define the <u>large-scale Lagrangian velocity</u>

$$\bar{\mathbf{v}}^t_{\ell,t_0}(\boldsymbol{\alpha}) = \tfrac{d}{dt}\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}) = \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}),t)$$

and <u>large-scale Lagrangian acceleration</u>

$$\bar{\mathbf{a}}^t_{\ell,t_0}(\boldsymbol{\alpha}) = \tfrac{d^2}{dt^2}\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}) = \overline{D}_{\ell,t}\bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}^t_{\ell,t_0}(\boldsymbol{\alpha}),t)$$

For simplicity hereafter we take $t_0 = 0$ and write $\bar{\mathbf{v}}_\ell(\boldsymbol{\alpha},t)$ and $\bar{\mathbf{a}}_\ell(\boldsymbol{\alpha},t)$ for the large-scale Lagrangian velocity &acceleration, respectively. By invoking the Navier-Stokes equation, one obtains

$$\bar{\mathbf{a}}_\ell(\boldsymbol{\alpha},t) = -\boldsymbol{\nabla}_x\bar{p}_\ell(\mathbf{x},t) + \mathbf{f}^s_\ell(\mathbf{x},t) + \nu\triangle\bar{\mathbf{u}}_\ell(\mathbf{x},t) + \bar{\mathbf{f}}^B_\ell(\mathbf{x},t)\Big|_{\mathbf{x}=\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t)}$$

We know from previous estimations that

$$\nu\triangle\bar{\mathbf{u}}_\ell = O(\nu\tfrac{\delta u(\ell)}{\ell^2}), \quad \bar{\mathbf{f}}^B_\ell = O(||\mathbf{f}^B||)$$

whereas

$$\boldsymbol{\nabla}\bar{p}_\ell, \;\; \mathbf{f}^s_\ell = O(\tfrac{\delta u^2(\ell)}{\ell}),^1$$

and the latter dominate at inertial-range scales. Thus, we conclude that

$$\bar{\mathbf{a}}_\ell(\boldsymbol{\alpha},t) = O(\tfrac{\delta u^2(\ell)}{\ell})$$

with $\delta u(\ell;\mathbf{x},t)$ evaluated at $\mathbf{x} = \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t)$. This gives a simple estimate of the large-scale Lagrangian velocity increment in time, using

[1] Note that for the pressure gradient this scaling is not established <u>locally</u>, but only in the sense of <u>space-average</u> of pth-powers. A useful local estimate is $\boldsymbol{\nabla}\bar{p}_\ell = O(\delta p(\ell)/\ell)$.

$$\delta\bar{\mathbf{v}}_\ell(\tau;\boldsymbol{\alpha},t) := \bar{\mathbf{v}}_\ell(\boldsymbol{\alpha},t+\tau) - \bar{\mathbf{v}}_\ell(\boldsymbol{\alpha},t) = \int_0^\tau \bar{\mathbf{a}}_\ell(\boldsymbol{\alpha},t+\sigma)d\sigma$$

Hence,

$$\delta\bar{\mathbf{v}}_\ell(\tau;\boldsymbol{\alpha},t) = O(\tfrac{\delta u^2_{\max}(\ell)}{\ell}\tau)$$

where $\delta u_{\max}(\ell) = \sup_{\sigma\in[0,\tau]} \delta u(\ell;\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t+\sigma),t+\sigma)$.

Since the natural time-scale of the Lagrangian velocity at length-scale ℓ is $\tau_\ell = \ell/\delta u(\ell)$, the local eddy turnover time, we may guess that the order of magnitude is the same for all $\sigma \in [0,\tau_\ell]$, i.e. $\delta u_{\max}(\ell) \cong \delta u(\ell)$ and thus, heuristically,

$$\delta\bar{\mathbf{v}}_\ell(\tau) = O^*(\tfrac{\delta u^2(\ell)}{\ell}\tau), \;\; \tau \lesssim \tau_\ell$$

In particular,

$$\delta\bar{\mathbf{v}}_\ell(\tau_\ell) = O^*(\delta u(\ell))$$

or, to good approximation,

$$\boxed{\delta\bar{\mathbf{v}}_\ell(\tau_\ell) \cong \delta\mathbf{u}(\ell).}$$

This result gives an important bridging relation between space-increments of the Eulerian velocity and the time-increments of the large-scale Lagrangian velocity.

It has furthermore been argued by

> G. Boffetta, F. De Lillo &S. Musacchio, "Lagrangian statistics and temporal intermittency in a shell model of turbulence," Phys. Rev. E. **66** 066307(2002)
>
> L. Biferale et. al.,"Multifractal statistics of Lagrangian velocity and acceleration in turbulence," Phys. Rev. Lett. **93** 064502 (2004)

that it should be true that

$$\delta\mathbf{v}(\tau_\ell) \cong \delta\bar{\mathbf{v}}_\ell(\tau_\ell)$$

where $\mathbf{v}(\boldsymbol{\alpha},t)$ is the full Lagrangian velocity from all scales of motion. We shall give a fairly careful argument for this which leads to a somewhat stronger conclusion that, pointwise and not just for increments,

$$\mathbf{v}(\boldsymbol{\alpha},t) = \bar{\mathbf{v}}_\ell(\boldsymbol{\alpha},t) + O(\delta u(\ell))$$

for $|t| \leq \tau_\ell$, where it is assumed that labeling is done at time $t_0 = 0$.

In the first place, we recall the result for the Eulerian velocity that

$$\mathbf{u}(\mathbf{x},t) - \bar{\mathbf{u}}_\ell(\mathbf{x},t) = \mathbf{u}'_\ell(\mathbf{x},t) = O(\delta u(\ell)),$$

which is the counterpart to the above Lagrangian result. We next compare the Lagrangian flows, $\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t)$ and $\mathbf{X}(\boldsymbol{\alpha},t)$, generated by the two velocity fields[2]. These satisfy

$$\mathbf{X}(\boldsymbol{\alpha},t) = \boldsymbol{\alpha} + \textstyle\int_0^t \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t'),t')dt',$$
$$\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t) = \boldsymbol{\alpha} + \textstyle\int_0^t \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t'),t')dt'$$

so that, taking the difference,

$$\mathbf{X}(\boldsymbol{\alpha},t) - \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t) = \textstyle\int_0^t \{\mathbf{u}'_\ell(\mathbf{X}(\boldsymbol{\alpha},t'),t') + [\bar{\mathbf{u}}_\ell(\mathbf{X}(\boldsymbol{\alpha},t'),t') - \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t'),t')]\}dt'$$

and thus

$$|\mathbf{X}(\boldsymbol{\alpha},t) - \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t)| \leq O(\delta u(\ell)t) + O(\tfrac{\delta u(\ell)}{\ell}) \textstyle\int_0^t |\mathbf{X}(\boldsymbol{\alpha},t') - \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t')|dt'$$

using $u'_\ell = O(\delta u(\ell))$, $\boldsymbol{\nabla}\bar{\mathbf{u}}_\ell = O(\frac{\delta u(\ell)}{\ell})$. If we only consider times $t \leq \tau_\ell = \frac{\ell}{\delta u(\ell)}$, then

$$\delta u(\ell)t = O(\ell).$$

We can then appeal to a standard mathematical result, the Gronwall inequality which states, in one simple form, that if

$$x(t) \leq a + b \textstyle\int_0^t x(s)ds$$

for all $t \in [0,T]$, then

$$x(t) \leq a \exp(bt)$$

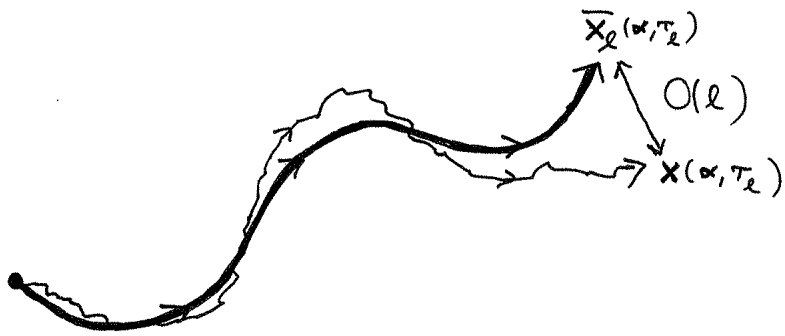
for $t \in [0,T]$. Applying this inequality we get

$$|\mathbf{X}(\boldsymbol{\alpha},t) - \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha},t)| \leq (const.)\ell \exp[O(\tfrac{\delta u(\ell)}{\ell}t)] = O(\ell)$$

for times $t \leq \tau_\ell = O(\ell/\delta u(\ell))$.

[2]It is important to stress that we are here considering $\mathbf{u}(\mathbf{x},t)$ to be the solution of the Navier-Stokes equation with $\nu > 0$. Even Leray singular solutions of NS are known to have sufficient regularity to define unique, volume-preserving flow maps, by a theorem of R. J. DiPerna & P. L. Lions, "Ordinary differential equations, transport theory and Sobolev spaces," Invent. Math. **98** 511-547 (1989). This was one of the works cited in the award to Lions of the Fields Medal in 1994.

Heuristically, since the difference in velocity of the two trajectories $\mathbf{X}(\boldsymbol{\alpha}, t), \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t)$ is $\mathbf{u}'_\ell = O(\delta u(\ell))$, they can differ over times $t \leq \tau_\ell$ by distances at most $O(\delta u(\ell) \cdot \tau_\ell) = O(\ell)$. The flow map $\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t)$ is a "smoothed" version of $\mathbf{X}(\boldsymbol{\alpha}, t)$:



The maximum distance up to time τ_ℓ is $O(\ell)$.

Finally, we compare the Lagrangian velocities, $\mathbf{v}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$ and $\bar{\mathbf{v}}_\ell(\boldsymbol{\alpha}, t) = \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t), t)$. Applying the previous result that $\mathbf{X}(\boldsymbol{\alpha}, t) - \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t) = O(\ell)$, we get for $t \leq \tau_\ell$ that

$$\mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) - \mathbf{u}(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t), t) = O(\delta u(\ell)).$$

Next we use again the Eulerian result that

$$\mathbf{u}(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t), t) - \bar{\mathbf{u}}_\ell(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t), t) = \mathbf{u}'_\ell(\overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t), t) = O(\delta u(\ell))$$

Putting this altogether, we conclude that

$$\mathbf{v}(\boldsymbol{\alpha}, t) = \bar{\mathbf{v}}_\ell(\boldsymbol{\alpha}, t) + O(\delta u(\ell))$$

for $t \leq \tau_\ell$, as claimed.

Since we may label particles at any arbitrary time t, we can conclude that

$$\begin{aligned} \delta\mathbf{v}(\tau_\ell; \boldsymbol{\alpha}, t) &= \mathbf{v}(\boldsymbol{\alpha}, t + \tau_\ell) - \mathbf{v}(\boldsymbol{\alpha}, t) \\ &= [\bar{\mathbf{v}}_\ell(\boldsymbol{\alpha}, t + \tau_\ell) - \bar{\mathbf{v}}_\ell(\boldsymbol{\alpha}, t)] + O(\delta u(\ell)) \\ &= \delta\bar{\mathbf{v}}_\ell(\tau_\ell; \boldsymbol{\alpha}, t) + O(\delta u(\ell)) \end{aligned} \tag{11}$$

However, we have argued earlier that

$$\delta\bar{\mathbf{v}}_\ell(\tau_\ell) \cong \delta\mathbf{u}(\ell).$$

We thus conclude that

$$\boxed{\delta\mathbf{v}(\tau_\ell) \cong \delta\mathbf{u}(\ell)}$$

This is the bridging relation between Lagrangian time-increments and Eulerian space-increments proposed by Boffetta et al. (2002) and further analyzed by Biferale et al. (2004).

We now examine some simple consequence of this relation, first within the perspective of K41 theory. Since in K41 $\tau_\ell \sim \langle\varepsilon\rangle^{-1/3}\ell^{2/3}$, one has that

$$(\langle\varepsilon\rangle\ell)^{1/3} \sim (\langle\varepsilon\rangle\tau)^{1/2}.$$

The standard K41 scaling $\langle(\delta u(\ell))^p\rangle \sim (\langle\varepsilon\rangle\ell)^{p/3}$ thus translates into

$$\text{K41:} \quad \langle(\delta v(\tau))^p\rangle \sim C_p(\langle\varepsilon\rangle\tau)^{p/2}$$

Such results go back, essentially, to the original paper of Kolmogorov in 1941. He proposed there that his similarity hypotheses could be applied to a general velocity increment of the form

$$\delta\mathbf{w}(\ell,\tau;\mathbf{x},t) = \mathbf{u}(\mathbf{x}+\ell+\mathbf{u}(\mathbf{x},t)\tau, t+\tau) - \mathbf{u}(\mathbf{x},t)$$

[with a slight change in notations]. For $\tau = 0, \delta\mathbf{w}(\ell,\tau=0)$ is the usual space-increment of velocity $\delta\mathbf{u}(\ell)$. On the other hand, one gets for $\ell = \mathbf{0}$ a "quasi-Lagrangian time-increment" following the fluid particle moving with the <u>initial</u> fluid velocity $\mathbf{u}(\mathbf{x},t)$. However, Kolmogorov did not work out the concrete predictions for Lagrangian velocity correlations. This seems to have been done first by Obukhov and by Landau, independently, just after the appearance of Kolmogorov's first paper in 1941. They both observed the $p = 2$ case of the above relation:

$$\langle(\delta v(\tau))^2\rangle \sim C_2\langle\varepsilon\rangle\tau$$

This result was first published, apparently, in the 1944 edition of the Landau &Lifshitz text on fluid mechanics. It was subsequently rediscovered by a number of people, in particular

E. Inoue, "On the turbulent diffusion in the atmosphere," J. Met. Soc. Japan **29** 246-252(1951)

E. Inoue, "On the Lagrangian correlation coefficient for turbulent diffusion and its application to atmospheric diffusion phenomena," Geophys. Research Papers **19** 397-412 (1951), Air Force Cambridge Research Laboratory

It is noteworthy that the linear scaling $\langle(\delta v(\tau))^2\rangle \propto \tau$ is identical to that for the time-increments of a Brownian motion/Wiener process, although the physics is quite different. One important consequence, however, is the same: just like the Wiener process, the Lagrangian velocity in turbulence in the limit $Re \to \infty$ is not differentiable in time! Instead, in K41 theory $\mathbf{v}(\boldsymbol{\alpha}, t)$ is Hölder continuous with (maximal) exponent 1/2 in the time variable t.

Another interesting historical sideline is that Richardson (1926) already raised similar issues. The title of his section 1.2 was "Does the wind possess a velocity?" He went on to explain:

> This question, at first sight foolish, improves on acquaintance. A velocity is defined, for example, in Lamb's "Dynamics" to this effect: Let Δx be the distance in the x direction passed over in a time Δt, then the x-component of velocity is the limit of $\Delta x/\Delta t$ as $\Delta t \to 0$. But for an air particle it is not obvious that $\Delta x/\Delta t$ attains a limit as $\Delta t \to 0$. We may really have to describe the position x of an air particle by something rather like Weierstrass's [continuous, nowhere-differentiable] function."

According to our modern understanding the Lagrangian velocity $\mathbf{v}(\boldsymbol{\alpha}, t) = d\mathbf{X}(\boldsymbol{\alpha}, t)/dt$ <u>does</u> exist in the infinite Reynolds number limit $Re \to \infty$, but the Lagrangian acceleration $\mathbf{a}(\boldsymbol{\alpha}, t) = d\mathbf{v}(\boldsymbol{\alpha}, t)/dt$ <u>does not</u> exist (at least in the classical sense) as $Re \to \infty$.

The previous results are all K41 style and ignore the possible effects of fluctuations. The first consideration of intermittency in Lagrangian statistics seems to have been given by

> M. S. Borgas, "The multifractal Lagrangian nature of turbulence," Phil. Trans. R. Soc. Lond. A **342** 379-411 (1993)

Borgas considered a description of intermittency based on energy dissipation. In that frame-

work, he proposed an analogue of the "bridging relation" $\delta v(\tau_\ell) \cong \delta u(\ell)$ with $\tau_\ell \cong \ell/\delta u(\ell)$. We shall here follow instead the discussion of Boffetta et al. (2002) and Biferale et al. (2004), which is instead in the spirit of the Parisi-Frisch theory for spatial intermittency of velocity increments. See also

> L. Chevillard et al.,"Lagrangian velocity statistics in turbulent flows: effects of dissipation," Phys. Rev. Lett. **91** 214502 (2003)

Following Boffetta et al. (2002), Biferale et al. (2004) let us then assume that

$$\delta v(\tau_\ell) \cong \delta u(\ell), \quad \tau_\ell \cong \frac{\ell}{\delta u(\ell)}$$

and also that

$$\delta u(\ell) \sim u_0 (\tfrac{\ell}{L})^h$$

at a given spacetime point with probability

$$\text{Prob}(\delta u \sim \ell^h) \sim (\tfrac{\ell}{L})^{\kappa(h)}$$

for a codimension spectrum $\kappa(h)$. From $\tau_\ell \cong \ell/\delta u(\ell)$ one then easily obtains that

$$\tfrac{\tau_\ell}{T} \sim (\tfrac{\ell}{L})^{1-h}, \;\; T \equiv \tfrac{L}{u_0} = \text{large-eddy turnover time}$$

It then follows that

$$\delta v(\tau) \cong \delta u(\ell) \sim u_0 (\tfrac{\ell}{L})^h \sim u_0 (\tfrac{\tau}{T})^{\frac{h}{1-h}}$$

and

$$\text{Prob}(\delta v \sim \tau^{h/(1-h)}) \sim (\tfrac{\tau}{T})^{\frac{\kappa(h)}{1-h}}.$$

Therefore,

$$\langle (\delta v(\tau))^p \rangle \sim u_0^p \int d\mu(h) (\tfrac{\tau}{T})^{\frac{ph+\kappa(h)}{1-h}}.$$

This yields by the usual steepest descent argument that

$$\langle (\delta v(\tau))^p \rangle \sim u_0^p (\tfrac{\tau}{T})^{\xi_p^L}, \quad \tau \ll T$$

with

$$\boxed{\xi_p^L = \inf_h[\tfrac{ph+\kappa(h)}{1-h}]. \qquad (*)}$$

This relation has a number of remarkable implications.

First, we note that $\kappa(h)$ can be recovered from the usual scaling exponents ζ_p of the space-increments of velocity by the inverse Legendre transform $D(h) = \inf_p[ph + (d - \zeta_p)]$ and $\kappa(h) = d - D(h)$. This in turn, by $(*)$, yields the exponents ξ_p^L. Thus, according to the multifractal theory, the exponents ζ_p and ξ_p^L are not independent but, in fact, are each uniquely derminable from the other! This is testable, parameter-free prediction.

Another interesting consequence of $(*)$ is that

$$\zeta_2^L = 1.$$

Thus, according to $(*)$, there is no intermittency correction to the Kolmogorov-Obukhov-Landau-Inoue relation $\langle(\delta v(\tau))^2\rangle \propto \tau$. This relation is analogous to the 4/5-law result that $\zeta_3 = 1$ for the exponents of space-increments. Not only are these analogous, but, in fact, they are equivalent within the multifractal model! One can see this as follows:

According to $\zeta_p = \inf_h[ph + \kappa(h)]$

$$\begin{aligned} & 1 = \zeta_3 = \inf_h[3h + \kappa(h)] \\ \iff \quad & \forall h, 1 \leq 3h + \kappa(h) \ \ \underline{\text{and}} \ \ \exists h_*, 1 = 3h_* + \kappa(h_*) \end{aligned} \tag{12}$$

Now, $1 \leq 3h + \kappa(h) \iff 1 \leq \frac{2h+\kappa(h)}{1-h}$ assuming that $h < 1$. Similarly,

$$1 = 3h_* + \kappa(h_*) \iff 1 = \tfrac{2h_*+\kappa(h_*)}{1-h_*}$$

Hence,

$$\begin{aligned} \zeta_3 = 1 \quad \iff \quad & \forall h, 1 \leq \frac{2h + \kappa(h)}{1-h} \ \ \underline{\text{and}} \ \ \exists h_*, 1 = \frac{2h_* + \kappa(h_*)}{1-h_*} \\ \iff \quad & 1 = \inf_h[\frac{2h + \kappa(h)}{1-h}] = \xi_2^L. \end{aligned} \tag{13}$$

Thus, within the multifractal theory,

$$\zeta_3 = 1 \iff \xi_2^L = 1.$$

This result is due to Boffetta, De Lillo & Musacchio (2002). It is noteworthy that lack of intermittency is found in the relations $\langle(\delta u_L(r))^3\rangle \propto \langle\varepsilon\rangle r$ and $\langle(\delta v(\tau))^2\rangle \propto \langle\varepsilon\rangle\tau$ in which the mean energy dissipation $\langle\varepsilon\rangle$ appears linearly. There is a general argument suggesting this should be so, due to R. H. Kraichnan, "On Kolmogorov's inertial-range theories," J. Fluid Mech. **62** 305-330(1974). See also UF, Section 6.4.2.

We now review some of the recent experimental and numerical evidence. DNS results have been presented by Biferale et al. (2004) and, also, by

L. Biferale et al.,"Particle trapping in three-dimensional fully developed turbulence," Phys. Fluids **17** 021701 (2005)

We reproduce Fig.2 from the latter paper, which shows structure functions of Lagrangian time-increments of velocity obtained from a 1024^3 DNS of forced, steady-state turbulence at $Re_\lambda = 284$. For exponents $p = 2, 4, 6$ it can be seen that the local slopes vary considerably and have no range where they are approximately constant. Thus, Biferale et al. (2005) employ the "extended self-similarity"(ESS) procedure of plotting

$$\frac{d(logS_p^L(\tau))}{d(\log S_2^L(\tau))} \ \ vs. \ \ \tau$$

rather than the local slope $d(\log S_p^L(\tau))/d(\log\tau)$. For more discussion of the ESS procedure, see UF, Section 8.3. We just note here that if the K-O-L-I relation $S_2^L(\tau) \propto \langle\varepsilon\rangle\tau$ holds, then these two plots will not differ in the inertial-range. The inset in Fig.2 shows that the ESS plot does show a narrow plateau for $p = 4, 6$ in the internal $[10\tau_\eta, 50\tau_\eta]$. Furthermore, the exponents fit from this range agree very well with the multifractal model prediction from formula $(*)$:

$$\xi_4^L/\xi_2^L = 1.7 \pm 0.05, \ \ \xi_6^L/\xi_2^L = 2.2 \pm 0.07.$$

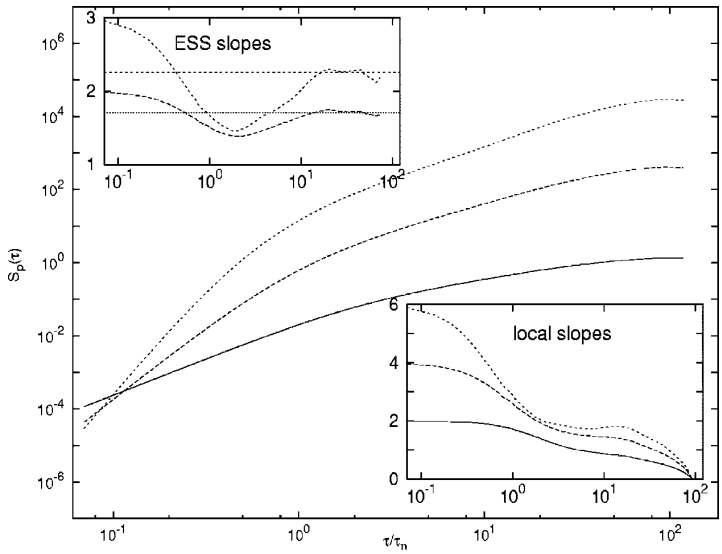


FIG. 2. Log–log plot of Lagrangian structure functions of orders $p=2$, 4, 6 (bottom to top) vs τ. Bottom right: logarithmic local slopes $d\log S_p(\tau)/d\log\tau$ (same line styles). Top left: ESS local slopes with respect to the second order structure function $d\log S_p(\tau)/d\log S_2(\tau)$, for $p=4$, 6 bottom and top, respectively. Straight lines correspond to the Lagrangian multifractal prediction with the same set of fractal dimensions used to fit the Eulerian statistics (Refs. 7 and 25). Data refer to the v_x component. The two other velocity components exhibit slightly worse scaling due to anisotropy effects. Relative scaling exponents and error bars are estimated from the mean and standard deviations of local slopes in the interval $[10\tau_\eta, 50\tau_\eta]$. Data refer to $R_\lambda=284$.

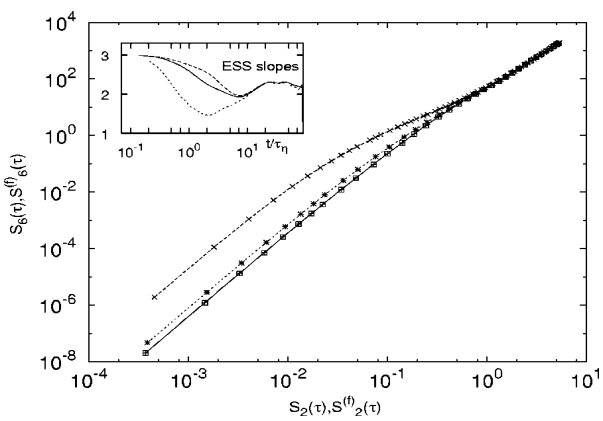


FIG. 3. ESS plots. Sixth-order structure functions vs the second-order one, with and without filtering of trapping events. Symbols refer to: $\times$ structure functions without any filtering, $S_p(r)$; $*$ structure function with filtering, $S_p^{(f)}(r)$, defined on a $\Delta_t=2\tau_\eta$ window; $\square$ with filtering on $\Delta_t=4\tau_\eta$. Inset: ESS local slopes of the curve in the body of the figure vs $\log(\tau/\tau_\eta)$. Upon filtering (two upper curves in the inset), the "bottleneck" effect on structure functions, i.e., the shallower slope observed in the neighborhood of τ_η, is suppressed. The behavior for time lags longer than $10\tau_\eta$ is unchanged. Data refer to $R_\lambda=284$. Similar results are obtained for structure function of order $p=4$ (not shown).

On the other hand, in the range from $[\tau_\eta, 10\tau_\eta]$ the exponents taken on rather smaller values

with local slopes implying a value

$$\xi_p^L \cong 2 \quad \text{for all} \quad p.$$

Biferale et al. (2005) explain this a consequence of "trapping" of Lagrangian particle trajectories, for times of that order, in the interior of intense, coherent vortices. By assuming that these events have $h_* = 0, D(h_*) = 1, \kappa(h_*) = 2$, they get $\xi_p^L = 2$ for all p. By "filtering out" the trapping events from the statistics, Biferale et al. (2005) in their Fig.3 find that the "dip" in the ESS plots is much reduced. For more details and discussion, see Biferale et al. (2005). Results from laboratory experiment are also available:

> N. Mordant et al., "Measurement of Lagrangian velocity in fully developed turbulence," Phys. Rev. Lett. **87** 214501 (2001); H. Xu et al.,"High-order Lagrangian velocity statistics," Phys. Rev. Lett. **96** 024503 (2006)

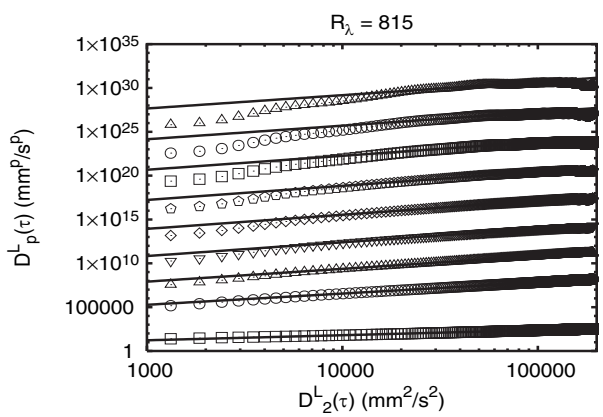


FIG. 2. ESS plot of the high-order Lagrangian structure functions at $R_\lambda = 815$. From top to bottom, the symbols correspond to our measurements of the tenth order through first order structure function, with second order omitted. The straight lines are fits to the data to extract the relative scaling exponents. The lines were fit only to values of $D_2^L(\tau)$ corresponding to times between $3\tau_\eta$ and $6\tau_\eta$, where $D_2^L(\tau)$ displayed a K41 scaling range with $\zeta_2^L \approx 1$.

TABLE I. Values of the relative scaling exponents measured in our experiment using ESS. The ESS curves were fit only in the range of times where the second order structure function displayed a K41 scaling range with exponent $\zeta_2^L \approx 1$. For comparison, we included the values measured from the DNS of Biferale *et al.* [8] and the experiment of Mordant *et al.* [9]

	R_λ	ζ_1^L/ζ_2^L	ζ_3^L/ζ_2^L	ζ_4^L/ζ_2^L	ζ_5^L/ζ_2^L	ζ_6^L/ζ_2^L	ζ_7^L/ζ_2^L	ζ_8^L/ζ_2^L	ζ_9^L/ζ_2^L	ζ_{10}^L/ζ_2^L
	200	0.59 ± 0.02	1.24 ± 0.03	1.35 ± 0.04	1.39 ± 0.07	1.40 ± 0.08	1.39 ± 0.09	1.40 ± 0.10	1.42 ± 0.11	1.46 ± 0.12
	690	0.58 ± 0.05	1.28 ± 0.14	1.47 ± 0.18	1.61 ± 0.21	1.73 ± 0.25	1.83 ± 0.28	1.92 ± 0.32	1.97 ± 0.35	1.98 ± 0.38
	815	0.58 ± 0.12	1.28 ± 0.30	1.47 ± 0.38	1.59 ± 0.46	1.66 ± 0.53	1.67 ± 0.60	1.65 ± 0.66	1.61 ± 0.73	1.57 ± 0.80
Ref. [8]	284			1.7 ± 0.05	2.0 ± 0.05	2.2 ± 0.07				
Ref. [9]	740	0.56 ± 0.01	1.34 ± 0.02	1.56 ± 0.06	1.73 ± 0.1	1.8 ± 0.2				

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Here we reproduce Fig. 2 of H. Xu et al. (2006), which gives the results for ESS plots of structure functions of Lagrangian time-increments of velocity obtained from a laboratory experiment of driven turbulence at $Re_\lambda = 815$ using optical tracking of Lagrangian particles. The corresponding exponents ξ_p^L, along with those of Mordant et al. (2001) and of Biferale et al. (2004, 2005), are given in their Table I, which is reproduced as well. It may be seen that the experimental results are considerably smaller than those obtained from DNS by Biferale et al. (2004, 2005). On the other hand, the experiments are more limited in the range of time-separations τ that they can study. The exponents of H. Xu et al. (2006), for example, are fit to data in the range from $3\tau_\eta$ to $6\tau_\eta$. If the DNS of Biferale et al. (2004, 2005) was employed in this same range it would yield exponents consistent with those from the experiments. The experimental results are thus consistent with the "trapping events" analyzed in detail by Biferale et al. (2005).

In addition,

H. Xu et al.,"Multifractal dimension of Lagrangian turbulence," Phys. Rev. Lett. **96** 114503 (2006)

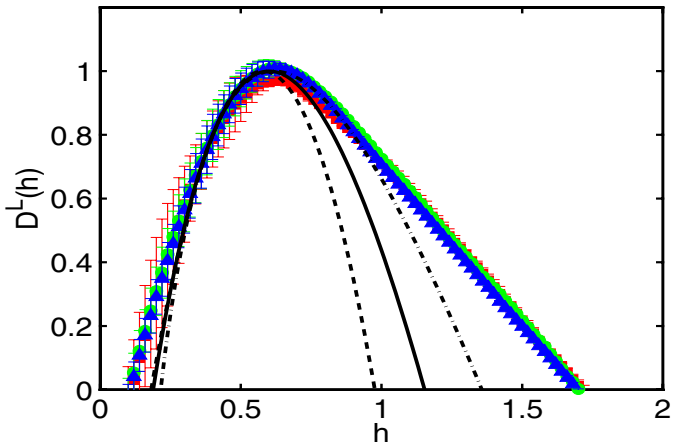

FIG. 3 (color online). Direct measurement of the Lagrangian multifractal dimension spectrum. The symbols denote our experimental measurements at three different Reynolds numbers: the ■ correspond to $R_\lambda = 200$, the ● to $R_\lambda = 690$, and the ▲ to $R_\lambda = 815$. The measured multifractal dimension spectra agree well for all three Reynolds numbers, suggesting that $D^L(h)$ has at most a weak Reynolds number dependence. The three curves correspond to models: the dashed line is the model due to Chevillard *et al.* [9], the solid line is Kolmogorov's log-normal model [13], and the dot-dashed line is the log-Poisson model of She and Lévêque [7].

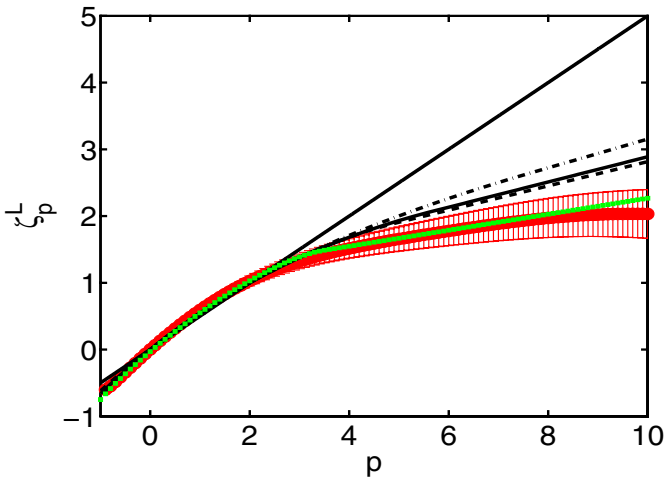

FIG. 4 (color online). Scaling exponents ζ_p^L of the Lagrangian structure functions as a function of order. The ● denote direct measurements of the $\zeta^L(p)$ at $R_\lambda = 690$. The ■ show the exponents extracted from our measured $D^L(h)$ data via Eq. (4). The two experimental measurements agree very well with each other. The curves are models: the dashed line is again the model of Chevillard *et al.* [9], the solid curved line is Kolmogorov's log-normal model [13], and the dot-dashed line is the model of She and Lévêque [7]. The solid straight line shows Kolmogorov's 1941 prediction for the ζ_p^L [12].

have attempted to obtain the Lagrangian multifractal spectrum $D^L(\hat{h})$ of the velocity time-increments, both directly and via the Legendre transform of ξ_p^L:

$$D^L(\hat{h}) = \inf_p[p\hat{h} + (d - \xi_p^L)]$$

Note that the relation $(*)$ gives, with $\kappa^L(\hat{h}) = d - D^L(\hat{h}),\ \kappa(h) = d - D(h)$,

$$\kappa^L(\hat{h}) = \tfrac{\kappa(h)}{1-h} = (1+\hat{h})\kappa\left(\tfrac{\hat{h}}{1+\hat{h}}\right)$$

with $\hat{h} = \frac{h}{1-h}$. Such a relation goes back to Borgas (1993). The direct measurements of Xu et al. (2006) for $D^L(\hat{h})$ are consistent with their measurements of ξ_p^L. Of course, as discussed above the experimental results for ξ_p^L (and thus also for $D^L(\hat{h})$) are consistently more singular than those predicted by $(*)$. Note also that Xu et al. cannot evaluate the multifractal spectrum for $h > h_{-1}$ corresponding to $p = -1$, since the usual structure functions diverge for $p < -1$. To access this portion of the multifractal spectrum, other techniques — such as inverse structure functions — are necessary.

We make finally some remarks about other forms of Lagrangian intermittency in fluid turbulence. It should be clear from our earlier discussion of Richardson 2-particle diffusion that it should also be subject to intermittency corrections. We found then that

$$\Delta^{(2)}(t) \cong (const.)t^{\frac{1}{1-h}}$$

when the velocity field has Hölder exponent h. It is easy to use this result to derive a multifractal generalization of the Richardson t^3-law, in the form

$$\langle[\Delta^{(2)}(t)]^p\rangle \sim L^p(\tfrac{t}{T_L})^{\mu_p}$$

with

$$\mu_p = \inf_h[\tfrac{p+\kappa(h)}{1-h}].$$

It is furthermore easy to show that

$$\zeta_3 = 1 \iff \mu_2 = 3,$$

so that the 4/5-law implies that

$$\langle[\Delta^{(2)}(t)]^2\rangle \sim \langle\varepsilon\rangle t^3$$

without any intermittency correction. All of these predictions are due to

G. Boffetta et al., "Pair dispersion in synthetic fully developed turbulence," Phys. Rev. E **60** 6734-6741(1999)

Of course, the test of these predictions will be difficult, since even Richardson's t^3-law has been very hard to verify in simulation or experiment. Just as there, it is easier to consider <u>inverse structure functions</u> or <u>exit statistics</u>, of the form

$$\langle[T_\lambda(\rho)]^p\rangle$$

for the λ-folding time $T_\lambda(\rho)$. It is particularly straightforward to consider <u>negative orders</u>, $p \to -p$, since

$$T_\lambda(\rho) \sim \rho^{1-h}$$

then implies in the multifractal model that

$$\langle[\tfrac{1}{T_\lambda(\rho)}]^p\rangle \sim (\tfrac{u_0}{L})^p(\tfrac{\rho}{L})^{\zeta_p - p} \quad (\star)$$

with the ζ_p's scaling exponents of the Eulerian velocity space-increments, $\zeta_p = \inf_h[ph + \kappa(h)]$. These predictions have been tested in DNS by

G. Boffetta and I. M. Sokolov, "Relative dispersion in fully developed turbulence: the Richardson's law and intermittency corrections," Phys. Rev. Lett. **88** 094501 (2002)

and also by Biferale et al. (2005). We reproduce Figure 7 from the latter paper, which seems to show better agreement of the DNS results with the multifractal prediction ($\star$) rather than with the K41 prediction $\propto \rho^{-2p/3}$.

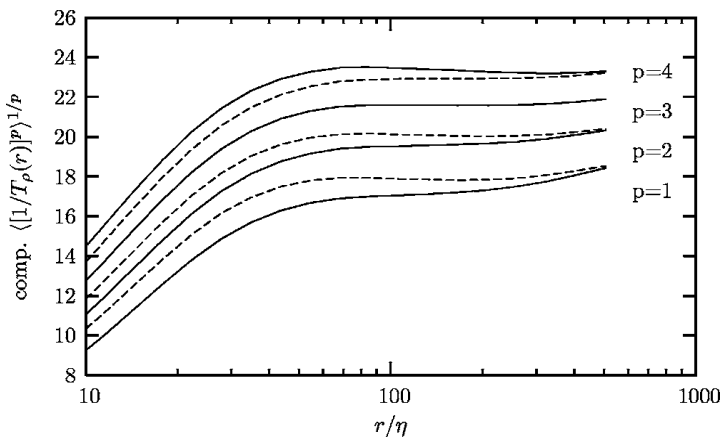

FIG. 7. The inverse exit time moments, $\langle[1/T_\rho(r)]^p\rangle^{1/p}$, for $p=1,\ldots,4$ compensated with the Kolmogorov scalings (solid lines) and the multifractal predictions (dashed lines) for the initial separation $r_0=1.2\eta$ and for $\rho=1.25$.

As we have discussed earlier, computer simulations have now advanced to the stage where direct comparison with Richardson's theory is possible. This extends to the direct study of intermittency effects. Consider the paper which we cited earlier for Richardson dispersion:

R. Bitane, H. Homann & J. Bec, "Geometry and violent events in turbulent pair dispersion," Journal of Turbulence, **14** 23–45 (2013)

Their Fig.4 (see below) plots the 4th and 6th-order moments of the relative separation versus time in their simulation with $Re_\lambda = 730$:

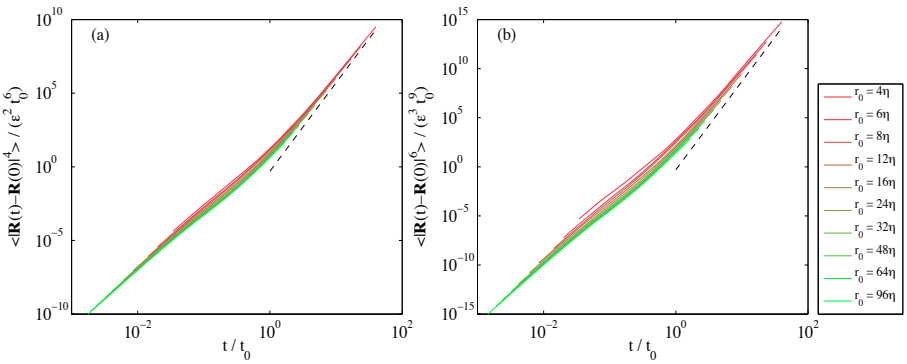

Figure 4. (a) Fourth-order moment $\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^4 \rangle$ and (b) sixth-order moment $\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^6 \rangle$ as function of t/t_0 for $R_\lambda = 730$. Both curves are normalized such that their expected long-time behavior is $\propto (t/t_0)^6$ and $\propto (t/t_0)^9$, respectively. The black dashed lines represent such behaviors.

Bitane et al. claim that the curves for different initial separations r_0 are well-described at long times by classical Richardson scaling with no intermittency corrections (black dashed lines). However, careful inspection shows that only the envelopes of these curves are parallel (approximately) to the dashed line. The individual curves have distinctly shallower slopes, consistent with sizable intermittency corrections!

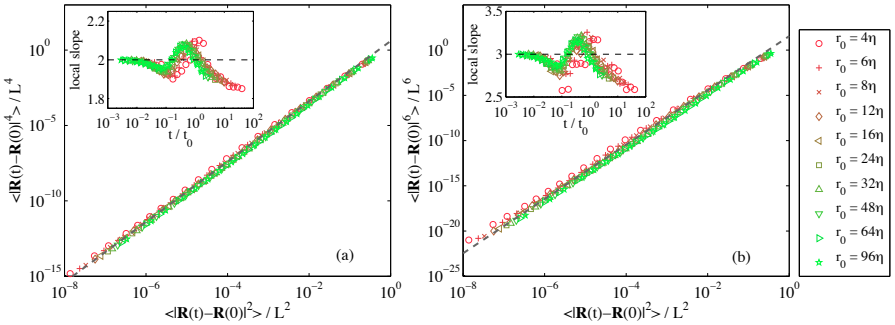

Figure 5. Fourth (a) and sixth (b) order moments of $|\boldsymbol{R}(t) - \boldsymbol{R}(0)|$ as a function of its second-order moment for $R_\lambda = 730$. The two gray dashed lines show a scale-invariant behavior, i.e. $\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^4 \rangle \propto \langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \rangle^2$ and $\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^6 \rangle \propto \langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \rangle^3$, respectively. The two insets show the associated local slopes, that is the logarithmic derivatives $\mathrm{d}\log\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^p \rangle / \mathrm{d}\log\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \rangle$, together with the normal scalings represented as dashed lines.

This is even more clear in Fig. 5 of Bitane et al., which plots relative scaling of the 4th and 6th-order moments versus the 2nd-order moments. Normal scaling would correspond to straight lines with constant slopes 2 and 3, respectively. However, the insets to the figures which plot local slopes of the individual curves show that straight lines are not great fits and local slopes are distinctly smaller than normal scaling values at long times.

As a last comment on the results of Bitane et al., they also examined the statistics of the relative velocities of Lagrangian particle pairs,

$$\mathbf{v}^{(2)}(t) = \frac{d\mathbf{R}^{(2)}}{dt} = \mathbf{v}(\boldsymbol{\alpha}', t) - \mathbf{v}(\boldsymbol{\alpha}, t), \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\rho}_0.$$

In particular they have considered the longitudinal component $v_{\parallel}^{(2)}(t)$ along the direction of $\mathbf{R}^{(2)}$, which satisfies

$$v_{\parallel}^{(2)}(t) = \frac{d\Delta^{(2)}}{dt}$$

with $\Delta^{(2)}(t) = |\mathbf{R}^{(2)}(t)|$. Using a multifractal model argument with $\Delta^{(2)}(t) \simeq (const.)t^{\frac{1}{1-h}}$ and thus $v_{\parallel}^{(2)}(t) = \frac{d\Delta^{(2)}}{dt} \simeq (const.)t^{\frac{h}{1-h}}$ would lead one to predict that the moments $\langle (v_{\parallel}^{(2)}(t))^p \rangle$ scale in time t the same as the (longitudinal) Lagrangian velocity structure-functions $\langle (\delta v(t))^p \rangle$, with exponents ξ_p^L given by the formula (*). This is exactly what Bitane et al. have verified for $p = 4$ and $p = 6$, as shown in their Fig.12!

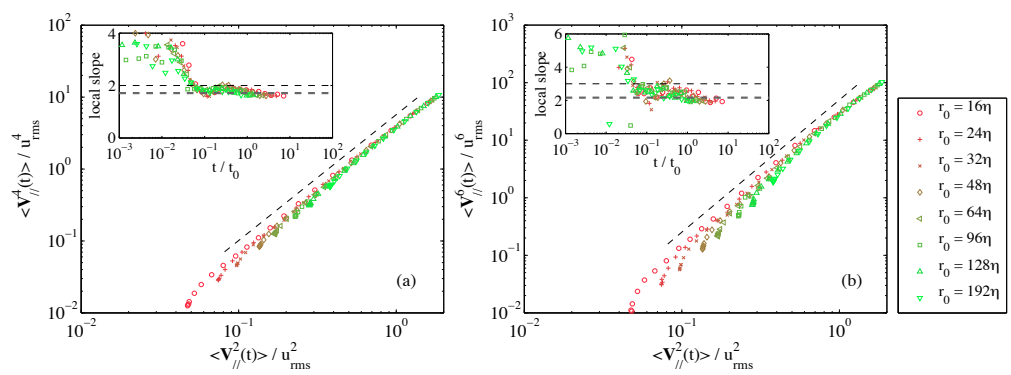

Figure 12. Fourth-order (a) and sixth-order (b) moments of the longitudinal velocity difference as a function of its second-order moment for various times and initial separations. The two dashed lines correspond to a scaling compatible with that of Lagrangian structure functions proposed in [25], namely $\zeta_4^{\rm L}/\zeta_2^{\rm L} = 1.71$ and $\zeta_6^{\rm L}/\zeta_2^{\rm L} = 2.16$. The insets show the logarithmic derivative $\mathrm{d}\log\langle [V^{\parallel}(t)]^p \rangle / \mathrm{d}\log\langle [V^{\parallel}(t)]^2 \rangle$ for (a) $p = 4$ and (b) $p = 6$ as a function of t/t_0; there the bold dashed lines show the Lagrangian multifractal scaling and the thin lines what is expected from a self-similar behavior.

Finally, we should note that there is also dissipation-range Lagrangian intermittency. For example, the small-time limit of the Lagrangian velocity increments is the Lagrangian acceleration:

$$\mathbf{a}(\boldsymbol{\alpha}, t) = \lim_{\tau \to 0} \frac{\delta \mathbf{v}(\tau; \boldsymbol{\alpha}, t)}{\tau} = \frac{d\mathbf{v}}{dt}(\boldsymbol{\alpha}, t)$$

which, from the Navier-Stokes equation, is given by

$$\mathbf{a}(\boldsymbol{\alpha}, t) = -\boldsymbol{\nabla}_x p(\mathbf{x}, t) + \nu \triangle \mathbf{u}(\mathbf{x}, t) + \mathbf{f}^B(\mathbf{x}, t)$$

evaluated at $\mathbf{x} = \mathbf{X}(\boldsymbol{\alpha}, t)$. Clearly, this quantity will be dominated by viscous effects. If we use the result

$$\bar{\mathbf{a}}_\ell(\boldsymbol{\alpha}, t) = -\boldsymbol{\nabla}_x \bar{p}_\ell(\mathbf{x}, t) + \nu \triangle \bar{\mathbf{u}}_\ell(\mathbf{x}, t) + \bar{\mathbf{f}}_\ell^B(\mathbf{x}, t)$$

evaluated at $\mathbf{x} = \overline{\mathbf{X}}_\ell(\boldsymbol{\alpha}, t)$ and the estimates

$$\boldsymbol{\nabla} \bar{p}_\ell, \mathbf{f}_\ell^s = O(\tfrac{\delta u^2(\ell)}{\ell}), \nu \triangle \bar{\mathbf{u}}_\ell = O(\tfrac{\nu \delta u(\ell)}{\ell^2})$$

then we see that the former balance the latter at the length scale such that $\ell \delta u(\ell) \cong \nu$ or

$$\eta_h \cong L Re^{-1/(1+h)}$$

at a point with Hölder exponent h. Alternatively, using

$$\tau_\ell = \tfrac{\ell}{\delta u(\ell)} \sim T_L (\tfrac{\ell}{L})^{1-h},$$

we see that this corresponds to a <u>fluctuating cut-off time scale</u>

$$\tau_h \cong T_L Re^{-(\frac{1-h}{1+h})}.$$

We can also estimate the acceleration itself locally as

$$a \cong \tfrac{\delta u^2(\eta_h)}{\eta_h} \cong \tfrac{u_0^2}{L} Re^{\frac{1-2h}{1+h}}, \quad Re \equiv \tfrac{u_0 L}{\nu}$$

where u_0 is the local (large-scale) fluctuating velocity. This line of reasoning has been used to developed a multifractal model of the <u>acceleration 1-point statistics</u> or acceleration PDF, by writing

$$a \cong \nu^{\frac{2h-1}{1+h}} u_0^{\frac{3}{1+h}} L^{\frac{-3h}{1+h}}$$

and then assuming a probability distribution of exponents h as $\nu \to 0$ distributed as $(\frac{\eta_h}{L})^{\kappa(h)}$ and a Gaussian distribution of u_0 with mean zero and variance $u_{rms}^2 = \langle u_0^2 \rangle$. For details, see

> L. Biferale et al., "Multifractal statistics of Lagrangian velocity and acceleration in turbulence," Phys. Rev. Lett. **93** 064502 (2004)

A comparison of this theory with DNS results shows quite satisfactory agreement, at least in the tails of the PDF:

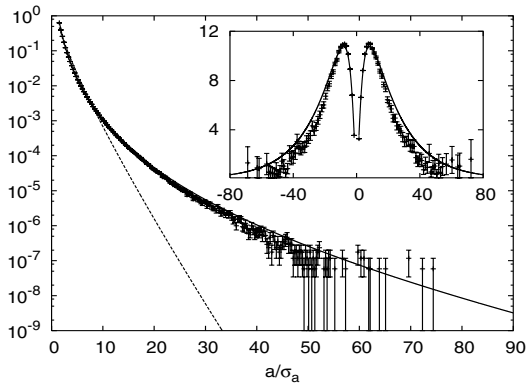

FIG. 2. Log-linear plot of the acceleration PDF. The crosses are the DNS data, the solid line is the multifractal prediction, and the dashed line is the K41 prediction. The DNS statistics were calculated along the trajectories of 2.0×10^6 particles amounting to 1.06×10^{10} events in total. The statistical uncertainty in the PDF was quantified by assuming that fluctuations grow like the square root of the number of events. Inset: $\tilde{a}^4 \mathcal{P}(\tilde{a})$ for the DNS data (crosses) and the multifractal prediction.