(B) 1-Particle and 2-Particle Turbulent Diffusion

The problem of 1-particle diffusion in a turbulent flow was first considered by

> G. I. Taylor, "Diffusion by continuous movements," Proc. Lond. Math. Soc. Series 2, **20** 196 (1921)

See also T&L, Section 7.1.

A question of great practical and theoretical interest is the mean square dispersion of particles, or

$$\langle|\delta\mathbf{X}(\boldsymbol{\alpha},t_0;t)|^2\rangle$$

where

$$\delta\mathbf{X}(\boldsymbol{\alpha},t_0;t)=\mathbf{X}(\boldsymbol{\alpha},t_0+t)-\mathbf{X}(\boldsymbol{\alpha},t_0)$$

is the displacement undergone by the particle between times t_0 and t_0+t. The average $\langle\cdot\rangle$ may be taken to be a volume-average over the particle positions $\boldsymbol{\alpha}$, or a time-average over the initial-time t_0, or both. It may also be taken to be an average over an ensemble of velocities. If the latter is homogeneous and stationary, then we may write $\langle|\delta\mathbf{X}(t)|^2\rangle$, since the average is independent of $\boldsymbol{\alpha},t_0$. Without loss of generality, let us take the labeling time to be $t_0=0$. In that case,

$$\begin{aligned}\delta\mathbf{X}(\boldsymbol{\alpha},t) &= \mathbf{X}(\boldsymbol{\alpha},t)-\boldsymbol{\alpha}\\ &= \int_0^t ds\ \mathbf{v}(\boldsymbol{\alpha},s)\end{aligned}\tag{4}$$

and

$$\langle|\delta\mathbf{X}(t)|^2\rangle=\int_0^t ds\int_0^t ds'\ \langle\mathbf{v}(s')\cdot\mathbf{v}(s)\rangle.$$

Let us assume, for simplicity, that the Lagrangian velocity process is stationary in time. This will be true, for example, if the turbulent ensemble is stationary and homogeneous, so that the statistics do not depend upon the particle's location or its past history. For a discussion of these issues, see T&L, Section 7.1. Under this assumption,

$$\langle\mathbf{v}(s')\cdot\mathbf{v}(s)\rangle=\langle\mathbf{v}(s'-s)\cdot\mathbf{v}(0)\rangle.$$

In that case, one can change the integration variables from s, s' to

$$\tau = s' - s, \quad T = \tfrac{1}{2}(s' + s)$$

with Jacobian of transformation $\left|\frac{\partial(\tau,T)}{\partial(s,s')}\right| = 1$. A bit of further calculation using $\langle \mathbf{v}(\tau)\cdot\mathbf{v}(0)\rangle = \langle \mathbf{v}(0)\cdot\mathbf{v}(-\tau)\rangle$ then gives

$$\langle|\delta\mathbf{X}(t)|^2\rangle = 2\int_0^t d\tau\ (t-\tau)\langle\mathbf{v}(\tau)\cdot\mathbf{v}(0)\rangle.$$

It follows that

$$\lim_{t\to\infty}\frac{\langle|\delta\mathbf{X}(t)|^2\rangle}{2t} = D$$

as long as

$$D = \int_0^\infty d\tau\langle\mathbf{v}(\tau)\cdot\mathbf{v}(0)\rangle \quad < +\infty \qquad (*)$$

As long as the latter condition holds, the long-time limit of the particle dispersion is diffusive, with $\langle|\delta\mathbf{X}(t)|^2\rangle \propto 2Dt$, like a Brownian motion. This is the basic result of Taylor (1921). In fact, it can be shown under the same condition $(*)$ more generally that

$$\lim_{\epsilon\to 0}\frac{\delta\mathbf{X}(t/\epsilon)}{\sqrt{2D/\epsilon}} = \mathbf{W}(t)$$

in the sense of distributions on path-space, where $\mathbf{W}(t)$ is d-dimensional Brownian motion.

The condition $(*)$ requires a rapid decay of time-correlations of the Lagrangian velocity $\mathbf{v}(\boldsymbol{\alpha},t)$. The relevant time-scale is the Lagrangian integral time-scale

$$\begin{aligned} T_L &= \frac{1}{\langle|\mathbf{v}(0)|^2\rangle}\int_0^\infty\langle\mathbf{v}(\tau)\cdot\mathbf{v}(0)\rangle d\tau \\ &= D/\langle|\mathbf{v}(0)|^2\rangle \end{aligned} \tag{5}$$

It is only for $t \gg T_L$ that $\delta\mathbf{X}(t)$ behaves diffusively. It is useful to note that the single-point statistics of the Lagrangian velocity $\mathbf{v}(\boldsymbol{\alpha},t)$ and the Eulerian velocity $\mathbf{u}(\mathbf{x},t)$ are the same, so that

$$\langle|\mathbf{v}(0)|^2\rangle = \langle|\mathbf{u}(0)|^2\rangle = u_{rms}^2.$$

For a detailed proof, see T&L, Section 7.1. One can argue on phenomenological grounds that

$$D \sim u_{rms}L,$$

where L is the integral length-scale, so that one gets

$$T_L = D/u_{rms}^2 \sim L/u_{rms},$$

the large-scale eddy turnover time. The Lagrangian time-correlation

$$C_L(\tau) = \langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle / u_{rms}^2$$

has been determined in a laboratory experiment with a "French washing machine" flow:

> N. Mordant et al., "Measurement of Lagrangian velocity in fully developed turbulence," Phys. Rev. Lett. **87** 214501 (2001)

and more recently in DNS of forced, homogeneous steady-state turbulence:

> L. Biferale et al., "Lagrangian statistics in fully developed turbulence," J. Turbulence **7** N0.6 (2006)

Both found that

$$C_L(\tau) \approx \exp(-\tau / T_L)$$

See the reproduced figure from the latter paper:

4 *L. Biferale et al.*

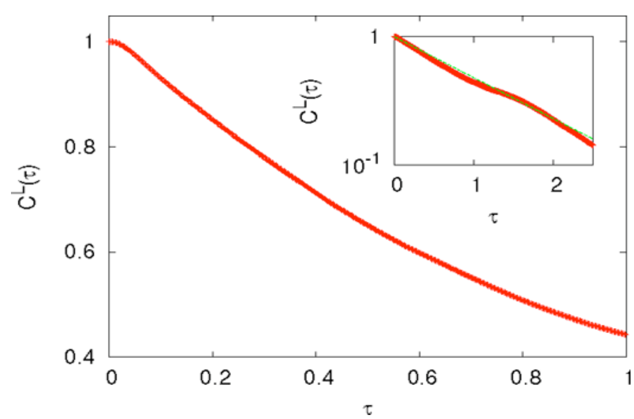

Figure 3. Lagrangian velocity autocorrelation function in linear coordinates and lin-log (inset) for the $R_\lambda = 284$ run. For comparison the exponential fit $\exp(-\tau / T_L)$ is also shown.

A more refined description of the 1-particle diffusion than the dispersion $\langle |\delta \mathbf{X}(\boldsymbol{\alpha}, t)|^2 \rangle$ is given by the 1-particle probability distribution

$$P_t^{(1)}(\mathbf{x}|\boldsymbol{\alpha}) = \langle \delta(\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{x}) \rangle$$

For homogeneous statistics this depends only on the displacement, $P_t^{(1)}(\mathbf{x}-\boldsymbol{\alpha}) = P^{(1)}(\mathbf{x}-\boldsymbol{\alpha}|\mathbf{0})$. At long times, $P_t^{(1)}(\mathbf{x}-\boldsymbol{\alpha}) \propto \exp\left(-|\mathbf{x}-\boldsymbol{\alpha}|^2/4Dt\right)/(4\pi Dt)^{d/2}$ as $t \to \infty$.

An important application of 1-particle diffusion is to the problem of evaluating the mean scalar evolution

$$\bar{\theta}(\mathbf{x},t) = \langle \theta(\mathbf{x},t) \rangle$$

Here we use the representation

$$\begin{aligned} \theta(\mathbf{x},t) &= \theta_0(\mathbf{A}(\mathbf{x},t)) \\ &= \int d^d\alpha\ \theta_0(\boldsymbol{\alpha})\delta(\mathbf{A}(\mathbf{x},t)-\boldsymbol{\alpha}) \end{aligned} \tag{6}$$

and the fact that

$$\delta(\mathbf{A}(\mathbf{x},t)-\boldsymbol{\alpha}) = \frac{\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x}))}{|\partial\mathbf{A}/\partial\mathbf{x}|} = \delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x})$$

since $|\partial\mathbf{A}/\partial\mathbf{x}| = 1$ by incompressibility. Thus,

$$\theta(\mathbf{x},t) = \int d^d\alpha\ \theta_0(\boldsymbol{\alpha})\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x}).$$

If we assume that the initial scalar field θ_0 is statistically independent of the velocity field, then the average factorizes as

$$\begin{aligned} \bar{\theta}(\mathbf{x},t) &= \int d^d\alpha\ \langle \theta_0(\boldsymbol{\alpha})\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x})\rangle \\ &= \int d^d\alpha\ \langle \theta_0(\boldsymbol{\alpha})\rangle\langle\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x}))\rangle \\ &= \int d^d\alpha\ \bar{\theta}_0(\boldsymbol{\alpha})P_t^{(1)}(\mathbf{x}|\boldsymbol{\alpha}). \end{aligned} \tag{7}$$

Thus, we see that $P_t^{(1)}$ propagates the mean scalar field forward in time.

2-particle diffusion

The problem of 2-particle turbulent diffusion was first considered by

> L. F. Richardson, "Atmospheric diffusion shown on a distance-neighbor graph," Proc. Roy. Soc. Lond. A **110** 709-737 (1926)

The basic quantity of interest is the 2-particle separation

$$\Delta^{(2)}(t) \equiv |\mathbf{X}(\boldsymbol{\alpha}', t) - \mathbf{X}(\boldsymbol{\alpha}, t)|, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}_0$$

which is the distance between two particles at time t which were initially displaced by $\Delta_0^{(2)} = |\Delta\boldsymbol{\alpha}_0|$. We start by deriving a rigorous estimate on $\Delta^{(2)}(t)$ under the assumption that the advecting velocity field is Hölder continuous with exponent h:

$$|\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)| \leq A|\mathbf{x}' - \mathbf{x}|^h.$$

By the reverse triangle inequality

$$\tfrac{d}{dt}|\mathbf{X}(\boldsymbol{\alpha}', t) - \mathbf{X}(\boldsymbol{\alpha}, t)| \leq \left|\tfrac{d}{dt}[\mathbf{X}(\boldsymbol{\alpha}', t) - \mathbf{X}(\boldsymbol{\alpha}, t)]\right|.$$

Thus,

$$\tfrac{d}{dt}\Delta^{(2)}(t) \leq |\mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}', t), t) - \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)|$$

using $(d/dt)\mathbf{X}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. Applying the Hölder continuity of the velocity then gives

$$\tfrac{d}{dt}\Delta^{(2)}(t) \leq A|\mathbf{X}(\boldsymbol{\alpha}', t) - \mathbf{X}(\boldsymbol{\alpha}, t)|^h = A[\Delta^{(2)}(t)]^h.$$

This simple differential inequality can easily be integrated to give a basic inequality

$$\Delta^{(2)}(t) \leq [\Delta_0^{(1-h)} + (1-h)A(t-t_0)]^{\frac{1}{1-h}}, \qquad (*)$$

If we assume that $|\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)| \cong A|\mathbf{x}' - \mathbf{x}|^h$, then we can further expect that the above inequality is an approximate <u>equality</u>

There is an important qualitative difference in the above estimate for $0 < h < 1$ and $h \to 1$. In the latter case, we can rewrite

$$\Delta^{(2)}(t) \leq \Delta_0 \left[1 + \tfrac{(1-h)A(t-t_0)}{\Delta_0^{1-h}}\right]^{\frac{1}{1-h}}$$

and we use $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$ to obtain, as <u>$h \to 1$</u>,

$$\Delta^{(2)}(t) \leq \Delta_0 \exp[A(t-t_0)].$$

This same estimate can also be obtained directly at $h = 1$. For $t \to \infty$, assuming near equality, we see that

$$\Delta^{(2)}(t) \cong \Delta_0 e^{A(t-t_0)}, \quad t \to \infty$$

so that <u>the initial separations is never forgotten.</u> On the contrary, for $0 < h < 1$,

$$\Delta^{(2)}(t) \cong [(1-h)A(t-t_0)]^{\frac{1}{1-h}}, \quad t \to \infty$$

and knowledge of Δ_0 is lost for long times.

In the Kolmogorov 1941 theory of turbulence, $h = 1/3$ in the inertial-range of turbulent flow, so that one can expect that $\Delta^{(2)}(t) \cong (t - t_0)^{3/2}$, or

$$[\Delta^{(2)}(t)]^2 \cong \langle \varepsilon \rangle (t - t_0)^3$$

in a dimensionally correct form. Of course, this asymptotics should apply only as long as $\eta \ll \Delta^{(2)}(t) \ll L$, i.e. at intermediate times. It was the prediction of Richardson (1926) that, indeed, the above scaling should hold in a mean sense

$$\langle [\Delta^{(2)}(t)]^2 \rangle \cong g_0 \langle \varepsilon \rangle (t - t_0)^3$$

with g_0 now called the Richardson constant. Notice that the growth of $\langle [\Delta^{(2)}(t)]^2 \rangle$ is much faster than "ballistic", i.e. faster than particles separating with a constant relative velocity U, which would lead to $[U(t - t_0)]^2$. The reason is that, as the particles separate, they experience larger relative velocities $\propto |x - x'|^{1/3}$.

The approach of Richardson (1926) to arrive at this result was quite different. He considered the statistics of 2-particle turbulent diffusion, which may be characterized by the 2-particle probability distribution

$$P_t^{(2)}(\mathbf{x}, \mathbf{x}' | \boldsymbol{\alpha}, \boldsymbol{\alpha}') = \langle \delta(\mathbf{X}(\boldsymbol{\alpha}, t) - \mathbf{x}) \delta(\mathbf{X}(\boldsymbol{\alpha}', t) - \mathbf{x}') \rangle$$

Setting $\mathbf{x}' = \mathbf{x} + \Delta\mathbf{x}$, $\boldsymbol{\alpha}' = \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}$ and assuming space homogeneity,

$$\begin{aligned} P_t^{(2)}(\mathbf{x}, \mathbf{x}' | \boldsymbol{\alpha}, \boldsymbol{\alpha}') &= P_t^{(2)}(\mathbf{x}, \mathbf{x} + \Delta\mathbf{x} | \boldsymbol{\alpha}, \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}) \\ &= P_t^{(2)}(\mathbf{x} - \boldsymbol{\alpha}, \mathbf{x} - \boldsymbol{\alpha} + \Delta\mathbf{x} | \mathbf{0}, \Delta\boldsymbol{\alpha}) \end{aligned} \tag{8}$$

Now let us form a reduced PDF just for the 2-particle separation

$$\bar{P}_t^{(2)}(\Delta\mathbf{x} | \Delta\boldsymbol{\alpha}) \equiv \int d^d\mathbf{x}\ P_t^{(2)}(\mathbf{x}, \mathbf{x} + \Delta\mathbf{x} | \mathbf{0}, \Delta\boldsymbol{\alpha}).$$

Since we know that for $t \gg t_0$, knowledge of $\Delta\boldsymbol{\alpha}$ is lost, let us also set

$$\bar{P}_t^{(2)}(\Delta\mathbf{x}) = \lim_{\Delta\boldsymbol{\alpha} \to \mathbf{0}} \bar{P}_t^{(2)}(\Delta\mathbf{x} | \Delta\boldsymbol{\alpha}).$$

If the velocity statistics are also isotropic and the orientation of $\Delta\boldsymbol{\alpha}$ is forgotten, then the PDF will depend only on the magnitude $\rho = |\Delta\mathbf{x}|$, i.e.

$$\bar{P}_t^{(2)}(\Delta\mathbf{x}) = \bar{P}_t^{(2)}(\rho).$$

Richardson (1926) hypothesized a diffusion equation for this quantity

$$\frac{\partial}{\partial t}\bar{P}_t^{(2)}(\rho) = \frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}[\rho^{d-1}K(\rho)\frac{\partial\bar{P}_t^{(2)}(\rho)}{\partial\rho}]$$

with a scale-dependent eddy-diffusivity

$$K(\rho) \cong \kappa_0\langle\varepsilon\rangle^{1/3}\rho^{4/3}.$$

Of course, Richardson's work was pre-Kolmogorov and he inferred $K(\rho) \propto \rho^{4/3}$ — rather miraculously — from a compilation of heterogeneous datasets on wind speeds from anemometers, separation rates of balloons in the atmosphere, and dispersion of volcanic ash! See the table in Richardson (1926), p. 724. Further, Richardson observed that this equation has an analytic solution for the initial condition

$$\bar{P}_{t=0}^{(2)}(\rho) = \delta(\rho)$$

of the form

$$\bar{P}_t^{(2)}(\rho) = \frac{A\rho^2}{(\kappa_0\langle\varepsilon\rangle^{1/3}t)^{9/2}}\exp\left[-\frac{9\rho^{2/3}}{4\kappa_0\langle\varepsilon\rangle^{1/3}t}\right]$$

with $A = (\frac{3}{2})^8/\Gamma(\frac{9}{2})$. From this solution various moments can be calculated, in particular,

$$\langle[\Delta^{(2)}(t)]^2\rangle = \langle\rho^2\rangle = g_0\langle\varepsilon\rangle t^3$$

with $g_0 = 1144\kappa_0^3/81$. Thus, again, the t^3-growth in time is obtained

Although these predictions are more than 90 years old, they have proved quite difficult to verify!

Richardson's t^3-law has been claimed to be observed in simple models of "synthetic turbulence" with a velocity field NOT governed by the Navier-Stokes equation, but with a similar Kolmogorov energy spectrum and scaling of velocity increments. E.g. consider

> F. W. Elliott, Jr. & A. J. Majda, "Pair dispersion over an inertial range spanning many decades," Phys. Fluids **8** 1052-1060(1996)

who, in a model of "synthetic turbulence" by Gaussian random fields, reported seeing the t^3 power-law over eight decades of time! However, it has been argued by

> D. J. Thomson & B. J. Devenish, "Particle pair separation in kinematic simulation," J. Fluid Mech. **526** 277-302 (2005).

that the observation of Elliott & Majda is a numerical artefact and not a feature of the actual model. More generally, Thomson & Devenish argue that the Richardson predictions should not hold in synthetic models of *Eulerian turbulence*, in which the field $\mathbf{u}(\mathbf{x}, t)$ represents the velocity observed at a fixed position $\mathbf{x}$ at time t. In such models, particle pairs are swept rapidly through non-moving eddies. This leads to a very short correlation time of the velocity increment, which suppresses the particle separation. Elliott & Majda chose a time-step in their numerical integration of the particle positions which was much too large to resolve this effect!

Note that in actual fluid turbulence governed by the Navier-Stokes equation, the small-scale eddies are swept together with the particles and thus the rapid decorrelation of velocity-increments is not expected to occur. Synthetic models of *Lagrangian turbulence* are designed to have the same property, by employing the model velocity in an equation for relative (not absolute) separations:

$$\frac{d}{dt}\Delta\mathbf{x} = \mathbf{u}_L(\Delta\mathbf{x}) \equiv \mathbf{u}(\Delta\mathbf{x}, t) - \mathbf{u}(\mathbf{0}, t).$$

The "quasi-Lagrangian velocity" $\mathbf{u}_L(\Delta\mathbf{x})$ can be interpreted as the velocity of the second particle at position $\Delta\mathbf{x}$ in a frame of reference moving with the velocity $\mathbf{u}(\mathbf{0}, t)$ of the first particle. Simulations with such models, e.g.

> G. Boffetta et al., "Relative dispersion in fully developed turbulence: Lagrangian statistics in synthetic flows," Europhys. Lett. **46**(2) 177-182 (1999)

observe not only Richardson's t^3-law but also Richardson's prediction for the stretched-exponential PDF of pair-separations. We shall see later that there is another model where Richardson's diffusion approximation is valid, when the velocity field has an extremely short-range in time.

Richardson's predictions have also proved difficult to verify in laboratory experiments. Consider two recent attempts:

> S. Ott and J. Mann, "An experimental investigation of the relative diffusion of particle pairs in three-dimensional turbulent flow," J. Fluid Mech. **422** 207-223(2000)

M. Bourgoin et al., "The role of pair dispersion in turbulent flow." Science **311** 835-838 (2006)

The first paper of Ott & Mann made some important theoretical contributions as well as carrying out experiments on grid-generated turbulence in water tanks. Although their experiments reached only $Re_\lambda \simeq 100$, their results were consistent with the predictions of Richardson. However, the second experiment in a "French washing machine" flow at $Re_\lambda \simeq 815$ did not observe Richardson's predictions. This is plausibly attributed to limitations on the initial distance Δ_0 between particles, which could not be taken smaller than about 30 Kolmogorov lengths. As can be seen from equation (*) for $h = 1/3$, it requires a time of order $\langle\varepsilon\rangle^{-1/3}\Delta_0^{2/3}$ to "forget" the initial separations. A new facility by the same group, see

`http://www.lfpn.ds.mpg.de/turbulence/tunnel.html`

is designed to reach $Re_\lambda \simeq 10^4$ and provide long enough times to forget initial separations.

Numerical simulations of Navier-Stokes turbulence have been more successful, in part because there are no limitations on the choice of Δ_0. For example, we consider the study of

L. Biferale et al."Lagrangian statistics of particle pairs in homogeneous isotropic turbulence," Phys. Fluids **17** 115101(2005)

In Figure 1 of that paper are plotted the results for $\langle\rho^3(t)\rangle$ from a 1024^3 DNS, $Re_\lambda \simeq 280$ with initial separations $\rho_0 = 1.2\eta, 2.5\eta, 9.8\eta$ and 19.6η. Clearly, the initial separations are never "forgotten". Furthermore, there is no clean t^3 range and no precise estimation of the Richardson constant g_0 is possible. There seem to be "crossover" effects from both the dissipation range and the energy range that prevent a clear verification of the inertial-range predictions. Longer inertial ranges seem necessary for this purpose. On the other hand, Richardson's predictions for $\bar{P}_t^{(2)}(\rho)$ seem to be quite well confirmed, even at the relatively low Reynolds numbers of the simulations. See Figure 2 in Biferale et al. (2005) for the comparison.

115101-3 Lagrangian statistics of particle pairs Phys. Fluids **17**, 115101 (2005)

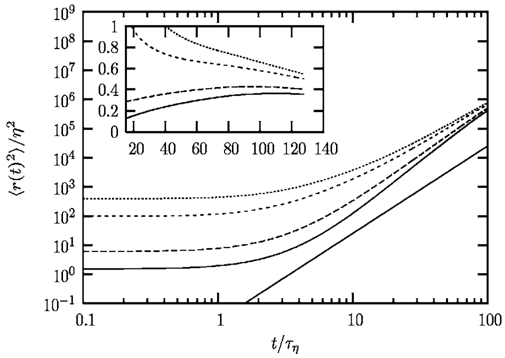

FIG. 1. The evolution of $\langle r(t)^2\rangle/\eta^2$ vs t/τ_η for the initial separations $r_0 = 1.2\eta$, $r_0 = 2.5\eta$, $r_0 = 9.8\eta$, and $r_0 = 19.6\eta$. The straight line is proportional to t^3. Inset: $\langle r(t)^2\rangle/\varepsilon t^3$ for the same four initial separations starting from $t/\tau_\eta \sim 15$.

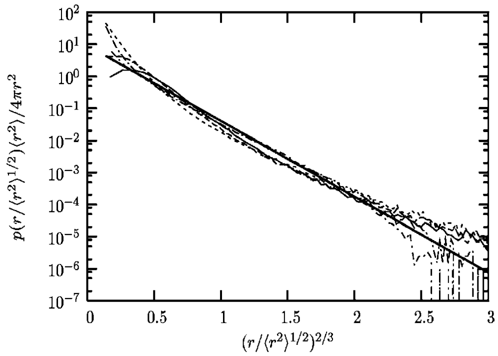

FIG. 2. Comparison of the Richardson PDF with the DNS data. The curves refer to data for $r_0 = 1.2\eta$ at $t = 5.2\tau_\eta$ (solid line), $t = 7\tau_\eta$ (long dashed line), $t = 14\tau_\eta$ (short dashed line), $t = 42\tau_\eta$ (dotted line), and $t = 70\tau_\eta$ (dot-dashed line). The thick solid line is the Richardson PDF (2).

To help disentangle the effects of different scales, an alternative approach has been proposed based on exit-time statistics. See

> G. Boffetta &I. M. Sokolov, "Relative dispersion in fully developed turbulence: The Richardson's law and intermittency corrections," Phys. Rev. Lett. **88** 094501 (2002)

In this approach one studies not $\rho(t) = \Delta^{(2)}(t)$ as function of time t but instead the λ-folding time (or exit time)

$$\begin{aligned} T_\lambda(\rho) &= \text{ first time for } \Delta^{(2)}(t) \text{ to increase from } \Delta^{(2)}(0) = \rho/\lambda \text{ to the distance } \rho \\ &= \sup\{t : \Delta^{(2)}(t) < \rho,\, \Delta^{(2)}(0) = \rho/\lambda\} \end{aligned} \tag{9}$$

for some $\lambda > 1$. This is analogous to what was done in going from structure functions to inverse structure functions in scaling of Eulerian velocity increments. The key advantage here is that this quantity focuses on the statistics at a fixed length-scale ρ. It may be shown using Richardson's equation that

$$\langle T_\lambda(\rho)\rangle = \frac{1}{2\kappa_0}\frac{\lambda^{2/3}-1}{\lambda^{2/3}}\frac{\rho^{2/3}}{\langle\varepsilon\rangle^{1/3}} \propto \rho^{2/3}.$$

Furthermore, the PDF of $T_\lambda(\rho)$ can be determined from

$$P_{\lambda,\rho}(T) = -\frac{d}{dT}\int_{|\Delta\mathbf{x}|<\rho} \bar{P}_T^{(2)}(\Delta\mathbf{x})d^3(\Delta\mathbf{x})$$

where $\bar{P}_T^{(2)}(\Delta\mathbf{x})$ is the solution of Richardson's equation with initial condition $\bar{P}_0^{(2)}(\Delta\mathbf{x}) = \frac{\lambda^2}{4\pi\rho^2}\delta(|\Delta\mathbf{x}| - \frac{\rho}{\lambda})$. This can be shown to lead to

$$P_{\lambda,\rho}(T) \sim \exp\left(-K\frac{\lambda^{2/3}-1}{\lambda^{2/3}}\frac{T}{\langle T_\lambda(\rho)\rangle}\right)$$

for $T \gg \langle T_\lambda(\rho)\rangle$ with $K \cong 2.72$ a numerical constant.

For all these results, see Boffetta &Sokolov (2002), Biferale et al. (2005), Biferale et al. (2006)

115101-5 Lagrangian statistics of particle pairs Phys. Fluids **17**, 115101 (2005)

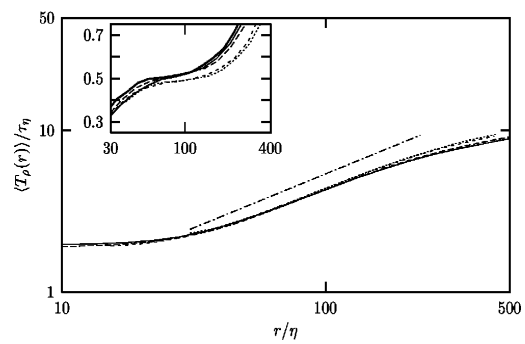

FIG. 5. The mean exit time for the initial separations $r_0=1.2\eta$ (thin continuous line), $r_0=2.5\eta$ (long dashed line), $r_0=9.8\eta$ (short dashed line), and $r_0=19.6\eta$ (dotted line) with $\rho=1.25$. The straight line is proportional to $r^{2/3}$. In the inset we show Richardson's constant, g, vs r/η as given by (9) for the same initial separations at $R_\lambda=284$. To evaluate the variability of g with the Reynolds number, we also plot a curve (thick continuous line) for the initial separation $r_0=1.2\eta$ at $R_\lambda=183$.

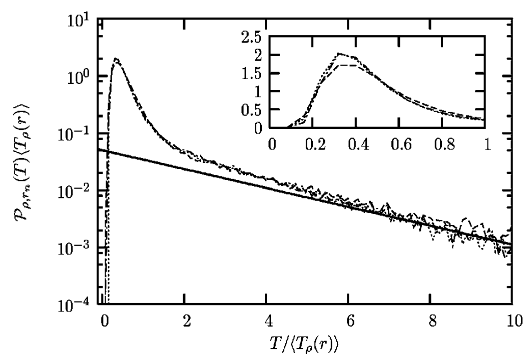

FIG. 6. The log-linear plot of the exit time PDF for $r_0=1.2\eta$ with $\rho=1.25$ at $r=21.8\eta$ (dashed line), $r=83.3\eta$ (dotted line), and $r=130.1\eta$ (dot-dashed line). The solid line is the large time prediction (10). Inset: a lin-lin plot of the same figure showing more detail.

In Figure 5 of Biferale et al. (2005) is plotted the result for $\langle T_\lambda(\rho)\rangle$, at a variety of initial separations ρ_0. There is now a much better collapse of results for different values of ρ_0, so that initial separations are "forgotten". Furthermore, there is a range of ρ with scaling very close to $\langle T_\lambda(\rho)\rangle \propto \rho^{2/3}$ [but with possibility a small intermittency correction that will be discussed later!]. The value of Richardson's constant can be inferred from

$$g_0 = \frac{143}{81}\frac{(\lambda^{2/3}-1)^3}{\lambda^2}\frac{\rho^2}{\langle\varepsilon\rangle\langle T_\lambda(\rho)\rangle}$$

which gives

$$g_0 \cong 0.50 \pm 0.05$$

in close agreement with the experimental determination of Ott &Mann(2000). In Figure 6 of Biferale et al. (2005) the prediction of Richardson's model for $P_{\lambda,\rho}(T)$ has also been shown to fit the DNS data quite well.

In the past decade, computer power has increased to the point that Richardson's predictions (with caveats) can be observed directly. For example, the following paper

> R. Bitane, H. Homann, and J. Bec, "Geometry and violent events in turbulent pair dispersion," Journal of Turbulence, **14** 23–45 (2013)

reports results of a 4096^3 DNS study at $Re_\lambda \simeq 730$:

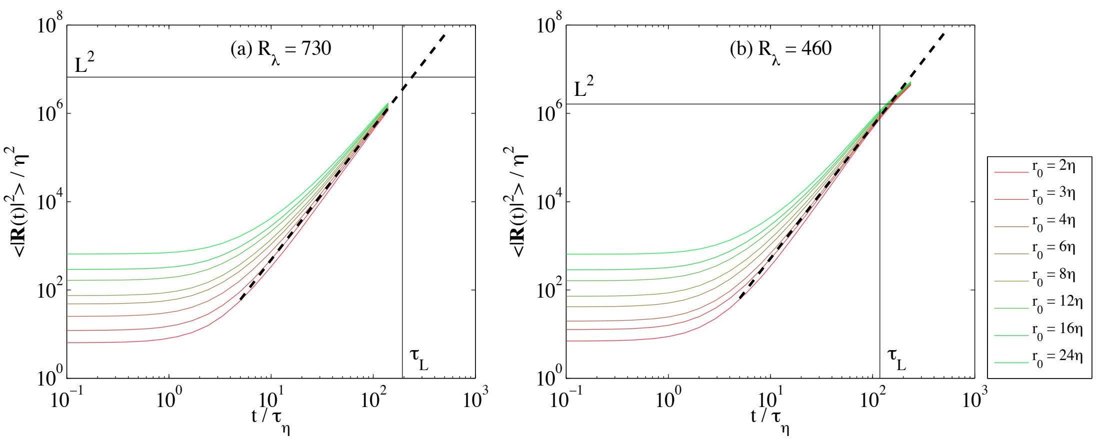

Figure 2. Time-evolution of the mean-squared distance for $R_\lambda = 730$ (a) and $R_\lambda = 460$ (b) for various initial separations r_0 as labeled. The horizontal and vertical solid lines represent the integral scale L and its associated turnover time τ_L, respectively. The dashed line corresponds to the explosive Richardson-Obukhov law (3) with $g = 0.52$.

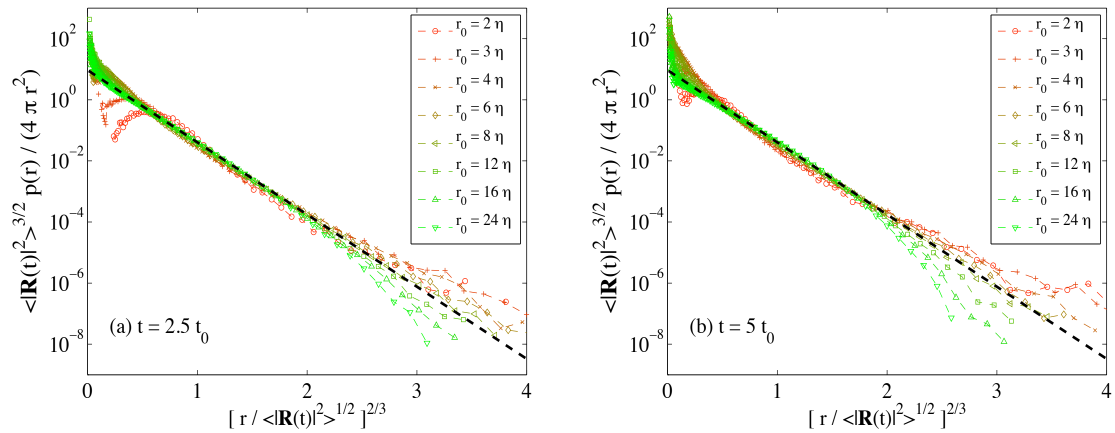

Figure 6. Probability density function of the distance r at time $t = 2.5\,t_0$ (a) and $t = 5\,t_0$ (b) and for various values of the initial separation. We have here normalized it by $4\pi r^2$ and represented on a $\log y$ axis as a function of $r/\langle|\boldsymbol{R}(t)|^2\rangle^{1/2}$. With such a choice, Richardson's diffusive density distribution (2) appears as a straight line (represented here as a black dashed line).

The t^3-law is directly observed, with clear "forgetting" of initial separations. The Richardson constant is determined to be $g = 0.52$, in good agreement with previous estimates. The Richardson stretched-exponential PDF of separation distances is also observed. Interestingly, however, there is agreement only for a limited range $1/2 \lesssim r/\langle r^2(t)\rangle \lesssim 2$. Outside this range there are clear deviations which remain to be understood! The diffusion approximation proposed by Richardson is clearly an *ad hoc* assumption and many interesting alternative models have been suggested. For example, see:

C. C. Lin, "On a theory of dispersion by continuous movements," Proc. Nat. Acad. Sci. USA 46, 566 (1960).

M. F. Shlesinger, B. J. West & Joseph Klafter, "Lévy dynamics of enhanced diffusion: Application to turbulence," Physical Review Letters **58** 1100 (1987).

G. L. Eyink & D. Benveniste, "Diffusion approximation in turbulent two-particle dispersion," Physical Review E **88** 041001 (2013)

S. Thalabard, G. Krstulovic & J. Bec, "Turbulent pair dispersion as a continuous-time random walk," Journal of Fluid Mechanics **755** R4 (2014).

The problem of 2-particle diffusion has an important relation with the evolution of 2-point statistics of passive scalars. Using the relation $\theta(\mathbf{x},t) = \int d^d\alpha\, \theta_0(\boldsymbol{\alpha})\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x})$ and assuming again that the <u>initial</u> scalar field is statistically independent of the velocity field, then

$$\begin{aligned} \langle \theta(\mathbf{x},t)\theta(\mathbf{x}',t)\rangle &= \int d^d\alpha \int d^d\alpha' \ \langle \theta_0(\boldsymbol{\alpha})\theta_0(\boldsymbol{\alpha}')\delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x})\delta(\mathbf{X}(\boldsymbol{\alpha}',t)-\mathbf{x}')\rangle \\ &= \int d^d\alpha \int d^d\alpha' \langle \theta_0(\boldsymbol{\alpha})\theta_0(\boldsymbol{\alpha}')\rangle\langle \delta(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{x})\delta(\mathbf{x}(\boldsymbol{\alpha}',t)-\mathbf{x}')\rangle \\ &= \int d^d\alpha \int d^d\alpha' \langle \theta_0(\boldsymbol{\alpha})\theta_0(\boldsymbol{\alpha}')\rangle P_t^{(2)}(\mathbf{x},\mathbf{x}'|\boldsymbol{\alpha},\boldsymbol{\alpha}'). \end{aligned} \tag{10}$$

Thus, $P_t^{(2)}$ evolves the 2-point correlation of the scalar in the same way as $P^{(1)}$ evolves the 1-point correlation. If the statistics of the velocity field and of the scalar field are both space-homogeneous, then

$$\langle \theta(\mathbf{x},t)\theta(\mathbf{x}',t)\rangle = \langle \theta(\mathbf{0},t)\theta(\Delta\mathbf{x},t)\rangle := \Theta^{(2)}(\Delta\mathbf{x},t), \quad \Delta\mathbf{x} = \mathbf{x}'-\mathbf{x},$$

and

$$P_t^{(2)}(\mathbf{x},\mathbf{x}'|\boldsymbol{\alpha},\boldsymbol{\alpha}') = P_t^{(2)}(\mathbf{y},\mathbf{y}+\Delta\mathbf{x}|\mathbf{0},\Delta\boldsymbol{\alpha}), \quad \Delta\boldsymbol{\alpha} = \boldsymbol{\alpha}'-\boldsymbol{\alpha},\ \mathbf{y} = \mathbf{x}-\boldsymbol{\alpha},$$

so that substitution into (10) gives

$$\Theta^{(2)}(\Delta\mathbf{x},t) = \int d^d(\Delta\alpha)\ \Theta_0^{(2)}(\Delta\boldsymbol{\alpha})\ \overline{P}_t^{(2)}(\Delta\mathbf{x}|\Delta\boldsymbol{\alpha})$$

where $\overline{P}_t^{(2)}(\Delta\mathbf{x}|\Delta\boldsymbol{\alpha}) = \int d^dy\ P_t^{(2)}(\mathbf{y},\mathbf{y}+\Delta\mathbf{x}|\mathbf{0},\Delta\boldsymbol{\alpha})$ is the reduced PDF. In particular at long times t when $|\Delta\mathbf{x}| \gg |\Delta\boldsymbol{\alpha}|$ and assuming statistical isotropy with $\rho = |\Delta\mathbf{x}|$, Richardson's diffusion equation $\partial_t\Theta^{(2)}(\rho,t) = \frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left[\rho^{d-1}K(\rho)\frac{\partial}{\partial\rho}\Theta^{(2)}(\rho,t)\right]$ should be reasonably accurate.

These latter results obviously generalize to N-particle diffusion described by transition probabilities $P_t^{(N)}(x_1,\cdots,x_N|\alpha_1,\cdots,\alpha_N)$ and N-point scalar correlations $\langle\theta(x_1,t)\cdots\theta(x_N,t)\rangle$. For more discussion of N-particle diffusion, see:

G. Falkovich, K. Gawędzki & M. Vergassola, "Particles and fields in fluid turbulence," Rev. Mod. Phys. **73** 913-975(2001)

L. Biferale et al., "Multiparticle dispersion in fully developed turbulence," Phys. Fluids **17** 111701 (2005)