V Lagrangian Dynamics & Mixing

See T&L, Ch. 7

The study of the Lagrangian dynamics of turbulence is, at once, very old and very new. Some of the earliest work on fluid turbulence in the 1920's, 30's and 40's, by pioneers such as Taylor, Richardson, and Kolmogorov, focused on Lagrangian aspects. It has long been recognized that this subject is of central importance in the field. The Lagrangian dynamics is intimately related to the process of mixing, which is both practically and theoretically important. The energy cascade itself is thought to be a fundamentally Lagrangian process.

However, the subject is also very new, with a dramatic increase of activity in recent years. This surge of interest has several causes. First, new theoretical progress has been made, especially in some exactly soluble models such as the Kraichnan model, proposed by him in 1968 but intensively studied in the 1990's. A second very important stimulus of recent activity is new experimental techniques and more powerful simulation facilities. There have — for the first time — opened up the Lagrangian dynamics of turbulence to precision empirical investigation. This intense study is still on-going and will hopefully yield dramatic new insights and progress!

(A) A Brief Review of Lagrangian Dynamics

The fundamental notion in Lagrangian dynamics is the idea of a Lagrangian flow $\mathbf{X}_{t_0}^t(\boldsymbol{\alpha})$, which maps the location $\boldsymbol{\alpha}$ of a particle at time t_0 to its location $\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}) = \mathbf{x}$ at time t. The variables $\boldsymbol{\alpha}$ are called the particle labels, which mark the particles according to their initial locations. (Note that other types of labels are possible.) The Lagrangian flow satisfies the equation, for advecting velocity $\mathbf{u}(\mathbf{x}, t)$:

$$\begin{cases} \frac{d}{dt}\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}) & = & \mathbf{u}(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}), t) \\ \mathbf{X}_{t_0}^{t_0}(\boldsymbol{\alpha}) & = & \boldsymbol{\alpha} \end{cases}$$

It is part of the standard theory of ordinary differential equations that, for a smooth velocity field $\mathbf{u}(\mathbf{x}, t)$, the Lagrangian maps $\boldsymbol{\alpha} \mapsto \mathbf{X}_{t_0}^t(\boldsymbol{\alpha})$ are diffeomorphisms of the flow domain. See, e.g.

V. I. Arnold, Ordinary Differential Equations, transl. from Russia by R. A. Silverman (MIT Press, Cambridge, MA, 1978)

That is, the maps are smooth functions of $\boldsymbol{\alpha}$ that are also invertible $\boldsymbol{\alpha} = (\mathbf{X}_{t_0}^t)^{-1}(\mathbf{x})$ with smooth inverse. Note, BTW, that $t > t_0$ and $t < t_0$ are both possible! The flow maps also satisfy the following group property

$$\mathbf{X}_{t'}^t \cdot \mathbf{X}_{t''}^{t'} = \mathbf{X}_{t''}^t, \quad \mathbf{X}_t^t = \mathbf{I}$$

This just expresses the property that the particle may be advected first from time t'' to t' and then from time t' to time t and the result will be the same as advecting directly from time t'' to t. If the advecting velocity is solenoidal, $\boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t) = 0$, then the flow maps will also be volume-preserving, which means that the Jacobian determinant is unity:

$$\left| \frac{\partial \mathbf{X}_{t_0}^t(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \right| = 1.$$

An important concept is the notion of the inverse Lagrangian map or the back-to-labels map

$$\mathbf{A}_{t_0}^t = (\mathbf{X}_{t_0}^t)^{-1}.$$

This maps the location $\mathbf{x}$ of the particle at time t back to its label $\boldsymbol{\alpha} = \mathbf{A}_{t_0}^t(\mathbf{x})$ at the initial time t_0. It is a consequence of the group property that

$$\mathbf{A}_{t_0}^t = \mathbf{X}_t^{t_0}.$$

Furthermore, taking the inverse of the group property for $\mathbf{A}_{t_0}^t$:

$$\mathbf{A}_{t''}^{t'} \circ \mathbf{A}_{t'}^t = \mathbf{A}_{t''}^t, \quad \mathbf{A}_t^t = \mathbf{I}.$$

Of course, if $\mathbf{u}$ is solenoidal, then

$$\left| \frac{\partial \mathbf{A}_{t_0}^t(\mathbf{x})}{\partial \mathbf{x}} \right| = \left| \frac{\partial \mathbf{X}_{t_0}^t(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \right|^{-1} = 1.$$

A dynamical equation can be derived for $\mathbf{A}_{t_0}^t$ by differentiating the relation

$$\mathbf{A}_{t_0}^t(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$$

and using the chain rule to obtain

$$\partial_t \mathbf{A}_{t_0}^t(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha})) + \mathbf{u}(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}), t) \cdot \boldsymbol{\nabla}_x \mathbf{A}_{t_0}^t(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha})) = 0$$

or

$$D_t\mathbf{A}_{t_0}^t(\mathbf{x}) = [\partial_t + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x]\mathbf{A}_{t_0}^t(\mathbf{x}) = 0$$

This equation just expresses the property that the labels $\boldsymbol{\alpha}$ are Lagrangian invariants and do not change in time along particle trajectories. This equation may be used to obtain the back-to-label maps by integrating in time from the initial condition $\mathbf{A}_{t_0}^{t_0}(\mathbf{x}) = \mathbf{x}$.

NOTE: If the initial time t_0 of labeling is understood, then the subscript t_0 is often omitted in the pair of functions $\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}), \mathbf{A}_{t_0}^t(\mathbf{x})$, which are simply written $\mathbf{X}(\boldsymbol{\alpha},t), \mathbf{A}(\mathbf{x},t)$. We shall occasionally do so, for simplicity.

Lagrangian dynamics & scalar mixing

There is a fundamental connection between Lagrangian dynamics and the advection of an ideal passive scalar, which satisfies

$$\partial_t\theta(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x\theta(\mathbf{x},t) = 0.$$

This equation can easily be solved by the method of characteristics as

$$\theta(\mathbf{x},t) = \theta(\mathbf{A}_{t_0}^t(\mathbf{x}), t_0),$$

using $D_t\mathbf{A}_{t_0}^t = 0$ and the chain rule. The equation

$$\theta(\mathbf{X}_{t_0}^t(\boldsymbol{\alpha}), t) = \theta(\boldsymbol{\alpha}, t_0)$$

just states that the scalar variable (temperature, dye concentration, etc.) is advected unchanged along fluid particle trajectories. Real passive scalars (non-ideal) are also subject to molecular diffusion, satisfying the advection-diffusion equation

$$\partial_t\theta(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x\theta(\mathbf{x},t) = \kappa\triangle\theta(\mathbf{x},t).$$

The method of characteristics can be adapted to this situation by considering stochastic particle trajectories which solve the stochastic ODE

$$d\widetilde{\mathbf{X}}_t^{t'}(\mathbf{x}) = \mathbf{u}(\widetilde{\mathbf{X}}_t^{t'}(\mathbf{x}), t') + \sqrt{2\kappa}\ d\mathbf{W}(t')$$

for a d-dimensional Brownian motion $\mathbf{W}(t') = (W_1(t'),\cdots,W_d(t'))$. Setting $\widetilde{\mathbf{A}}_{t_0}^t(\mathbf{x}) \equiv \widetilde{\mathbf{X}}_t^{t_0}(\mathbf{x})$, one obtains

$$\theta(\mathbf{x},t) = \mathbb{E}[\theta(\widetilde{\mathbf{A}}_{t_0}^t(\mathbf{x}), t_0)]$$

where $\mathbb{E}[\cdot]$ is the average over the realizations of the Brownian motion. For a review of this

Lagrangian technique to model molecular diffusion, see, for example,

B. Sawford, "Turbulent Relative Dispersion", Annu. Rev. Fluid Mech. **33** 289–317 (2001)

or, for a more mathematical perspective on stochastic method of characteristics,

H. Kunita, Stochastic Flows & Stochastic Differential Equations (Cambridge University Press, Cambridge, 1990), Theorem 6.2.5

Lagrangian formulation of fluid dynamics

The discussion to this point has been valid for a general advecting velocity $\mathbf{u}(\mathbf{x},t)$, not necessarily a solution of the fluid equations. However, the Navier-Stokes dynamics [and other fluid equations] can be formulated in terms of the Lagrangian flow map $\mathbf{X}^t_{t_0}(\boldsymbol{\alpha})$. Note that

$$\begin{aligned}\frac{d}{dt}\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}) &= \mathbf{u}(\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}),t) \\ &\equiv \mathbf{v}^t_{t_0}(\boldsymbol{\alpha}) \qquad (1)\end{aligned}$$

which is the so-called Lagrangian velocity. If t_0 is understood one may simply write $\mathbf{v}(\boldsymbol{\alpha},t)$ for $\mathbf{v}^t_{t_0}(\boldsymbol{\alpha})$. Now note that

$$\tfrac{d^2}{dt^2}\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}) = \tfrac{d}{dt}\mathbf{v}^t_{t_0}(\boldsymbol{\alpha}) = (D_t\mathbf{u})(\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}),t)$$

by the chain rule, with $D_t = \partial_t + \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}_x$. Using the Navier-Stokes equation for $D_t\mathbf{u}$, we thus obtain

$$\tfrac{d^2}{dt^2}\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}) = -\boldsymbol{\nabla}p(\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}),t) + \nu\triangle\mathbf{u}(\mathbf{X}^t_{t_0}(\boldsymbol{\alpha}),t)$$

which is the Lagrangian equation of motion. This is a mathematically quite complicated equation since $\boldsymbol{\nabla}p(\mathbf{x},t)$ and $\nu\triangle\mathbf{u}(\mathbf{x},t)$ contain Eulerian derivatives. Physically, it is just Newton's equation, $\mathbf{F}=m\mathbf{a}$, for individual fluid particles.

The inviscid case ($\nu=0$) corresponding to the incompressible Euler equations has special properties due to the fact that the dynamics is Hamiltonian. For example, the Lagrangian equation of motion follows from a least-action principle, which states that the particle trajectories minimize the action

$$S_{t_0}^t[\mathbf{x};\boldsymbol{\alpha}] = \int_{t_0}^t d\tau \left[|\dot{\mathbf{X}}(\boldsymbol{\alpha},\tau)|^2 - p(\mathbf{X}(\boldsymbol{\alpha},t),t)\,(1-\det(\boldsymbol{\nabla}_\alpha \mathbf{X}(\boldsymbol{\alpha},t)))\right]$$

individually for each fluid particle $\boldsymbol{\alpha}$. For a readable discussion of Hamiltonian theory, see:

R. Salmon, "Hamiltonian fluid mechanics," Ann. Rev. Fluid Mech. **20** 225-256 (1988)

The Hamiltonian formulation reveals many important symmetries and conservation properties of the Euler equations which have a deep basis in the Lagrangian description of the fluid.

The vorticity, in particular, has fundamental Lagrangian properties. One way to express these is through the so-called Cauchy invariant

$$\Omega_i \equiv \omega_j(\mathbf{x},t)\frac{\partial A_i}{\partial x_j}(\mathbf{x},t)$$

with $\mathbf{A}(\mathbf{x},t) := \mathbf{A}_{t_0}^t(\mathbf{x})$ for fixed t_0, or

$$\boldsymbol{\Omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla}_x)\mathbf{A}$$

shown by A. L. Cauchy to be a Lagrangian invariant

$$D_t\boldsymbol{\Omega} = \mathbf{0}.$$

To demonstrate this, we first observe the following equation for the gradient of the inverse Lagrangian map:

$$D_t(\boldsymbol{\nabla}_x\mathbf{A}) + (\boldsymbol{\nabla}_x\mathbf{u})(\boldsymbol{\nabla}_x\mathbf{A}) = \mathbf{0},$$

which is easily proved by taking the gradient of the label-invariance equation $D_t\mathbf{A} = \mathbf{0}$. It then follows that

$$\begin{aligned} D_t\boldsymbol{\Omega} &= D_t\boldsymbol{\omega}\cdot\boldsymbol{\nabla}_x\mathbf{A} + \boldsymbol{\omega}\cdot D_t(\boldsymbol{\nabla}_x\mathbf{A}) \\ &= [\boldsymbol{\omega}\cdot\boldsymbol{\nabla}_x\mathbf{u}]\cdot\boldsymbol{\nabla}_x\mathbf{A} + \boldsymbol{\omega}\cdot[-(\boldsymbol{\nabla}_x\mathbf{u})(\boldsymbol{\nabla}_x\mathbf{A})] \\ &= \mathbf{0}. \end{aligned} \tag{2}$$

This implies that the Cauchy invariant is a time-independent function $\boldsymbol{\Omega}(\boldsymbol{\alpha})$ of the particle label. Using the fact that $(\boldsymbol{\nabla}_x\mathbf{A})^{-1} = (\boldsymbol{\nabla}_{\boldsymbol{\alpha}}\mathbf{X})$, one derives the Cauchy vorticity formula

$$\boldsymbol{\omega}(\mathbf{x}(\boldsymbol{\alpha},t),t) = \boldsymbol{\Omega}(\boldsymbol{\alpha})\cdot\boldsymbol{\nabla}_{\boldsymbol{\alpha}}\mathbf{X}(\boldsymbol{\alpha},t)$$

which concisely expresses the Lagrangian properties of the vorticity. Of course, $\mathbf{\Omega}(\boldsymbol{\alpha})$ is just the initial vorticity $\boldsymbol{\omega}(\boldsymbol{\alpha}, t_0)$ at the time of labeling. We derived this formula of Cauchy in the previous chapter from a slightly different point of view.

This equation has a very simple physical interpretation. Consider an initial infinitesimal displacement, or line element, $\delta\boldsymbol{\alpha}$ between two points $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} + \delta\boldsymbol{\alpha}$. Materially advected by the fluid, this element evolves into

$$\begin{aligned} \delta\mathbf{X}(\boldsymbol{\alpha}, t) &\equiv \mathbf{X}(\boldsymbol{\alpha} + \delta\boldsymbol{\alpha}, t) - \mathbf{X}(\boldsymbol{\alpha}, t) \\ &= \delta\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}_{\boldsymbol{\alpha}}\mathbf{X}(\boldsymbol{\alpha}, t). \end{aligned} \tag{3}$$

This is the same form as the Cauchy formula for the vorticity. Thus, Cauchy's formula just states the fundamental property of the vorticity vector that it evolves like an infinitesimal material line element. It is not difficult to show that this result is equivalent to similar Lagrangian invariance properties, such as the Kelvin-Helmholtz theorem. For more detailed discussion, see either the review article of Salmon or treatises on vortex dynamics, such as

P. G. Saffman, Vortex Dynamics (Cambridge University Press, Cambridge, 1992)

J.-Z. Wu, H.-Y. Ma and M. -D. Zhou, Vorticity and Vortex Dynamics (Springer, Berlin, 2006)

Finally, note that the Constantin-Iyer (2008) theorem on stochastic representation of solutions to the viscous Helmholtz equation provides a generalization of the Cauchy invariant to the incompressible Navier-Stokes equation. Since $\tilde{\mathbf{X}}_{t_0}^t = \tilde{\mathbf{A}}_t^{t_0}$, that representation can be written as

$$\boldsymbol{\omega}(\mathbf{x}, t) = \overline{\boldsymbol{\omega}(\boldsymbol{\alpha}, t_0) \cdot \boldsymbol{\nabla}_\alpha \tilde{\mathbf{X}}_{t_0}^t(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\tilde{\mathbf{A}}_{t_0}^t(\mathbf{x})}} = \overline{\boldsymbol{\omega}(\boldsymbol{\alpha}, t_0) \cdot \boldsymbol{\nabla}_\alpha \tilde{\mathbf{A}}_t^{t_0}(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\tilde{\mathbf{X}}_t^{t_0}(\mathbf{x})}},$$

so that

$$\frac{d}{dt_0}\overline{\boldsymbol{\omega}(\boldsymbol{\alpha}, t_0) \cdot \boldsymbol{\nabla}_\alpha \tilde{\mathbf{A}}_t^{t_0}(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\tilde{\mathbf{X}}_t^{t_0}(\mathbf{x})}} = \mathbf{0}.$$