(F) Possible Power-Laws

The previous section presented the "traditional theory" of high-Reynolds number scaling in channel and pipe flow, as in Tennekes & Lumley, Section 5.2. It should be clear, however, that no rigorous, a priori deductions were made there. Results such as the log-law were obtained from precisely stated assumptions which were shown to be consistent with the RANS equations (or, at least, with a subset of them!) Other assumptions may also be possible and lead to different results, equally consistent with the fluid equations.

There is a long tradition of modeling the mean velocity field $\bar{u}(y)$ by a power-law rather than by a logarithm. For example, see

> L. Prandtl, "The mechanics of viscous fluids," in Aerodynamics Theory, W. F. Durand, ed. (Julius Springer, Berlin, 1935), vol. 3, pp. 34-208.

or

> Modern Developments in Fluid Dynamics, ed. S. Goldstein (Clarendon Press, Oxford, 1938), vol. II, Section VIII. 155, pp. 339-340.

Prandtl noted that the power-law

$$\bar{u}(y)/u_* = 8.7(y^+)^{1/7}$$

gave a good match with measured velocity profiles in pipe-flow over the entire radius of the pipe, for Reynolds number less than about 10^5. It is not hard to show that this mean-velocity leads to a power-law friction-law, the so-called Blasius resistance formula for the friction coefficient $\lambda = 2u_*^2/\bar{u}_m^2$

$$\lambda = 0.0665(Re_*)^{-1/4},$$

also in good agreement with experiment for $Re_* \lesssim 10^5$. However, Prandtl noted that agreement with experiment for larger Reynolds number required a form

$$\bar{u}(y)/u_* = C_\alpha (y^+)^\alpha + B_\alpha$$

with $\alpha \to 0$ as $Re_* \to \infty$. Prandtl speculated that the true mean profile might have such a power-law form, approaching the logarithmic profile as $Re_* \to \infty$.

More recently, this old idea of Prandtl has been revived by G. I. Barenblatt and his collaborators. For example, see:

G. I. Barenblatt, "Scaling law for fully developed turbulent shear flows. Part I. Basic hypotheses and analysis," J. Fluid Mech. **248** 513-520 (1993)

G. I. Barenblatt & A. J. Chorin, "Small viscosity asymptotics for the inertial range of local structure and for the wall region of wall-bounded turbulent shear flow," Proc. Nat. Acad. Sci. **93** 6749-6752 (1996)

G. I. Barenblatt & A. J. Chorin, "Scaling laws and vanishing viscosity limits for wall-bounded shear flows and for local structure in developed turbulence," Comm. Pure. Appl. Math. vol. L 381-398(1997)

G. I. Barenblatt, A. J. Chorin & V. M. Prostokishin, "Scaling laws for fully developed turbulent flow in pipes," Appl. Mech. Rev. **50** 413-429 (1997)

and many others! These papers connected the power-law velocity profile with the notion of incomplete similarity. Let us say a few words about this here. We have already seen a good example, when we discussed the velocity structure-functions and small-scale intermittency. Dimensional analysis then gives

$$\langle (\delta u_L(r))^p \rangle = (\bar{\varepsilon} r)^{p/3} F_p(\tfrac{r}{L})$$

if one assumes that the zero-viscosity limit $\nu \to 0$ exists. If one furthermore assumes that

$$\lim_{x\to 0} F_p(x) = C_p < +\infty$$

then the K41 predictions are obtained at small-scales

$$\langle (\delta u_L(r))^p \rangle \sim C_p (\bar{\varepsilon} r)^{p/3}, \quad \eta \ll r \ll L.$$

This (if it were true) would be an example of so-called <u>complete similarity</u>. However, as we have discussed earlier, there is considerable evidence that instead

$$F_p(x) \sim C_p x^{\delta\zeta_p}, \quad x \ll 1$$

with $\delta\zeta_p > 0$ for $p > 3$, and thus that

$$\langle(\delta u_L(r))^p\rangle \sim C_p(\bar{\varepsilon}r)^{p/3}(\tfrac{r}{L})^{\delta\zeta_p}, \quad \eta \ll r \ll L.$$

This type of scaling is what Barenblatt terms incomplete similarity. It is what is known to occur in Burgers turbulence ("Burgulence") and the Kraichnan model of the passive scalar. There are other examples in physics where this type of phenomenon occurs. For example, in equilibrium spin-systems in d-dimensions, the correlation function of the magnetization $m(\mathbf{x})$ is given by dimensional analysis as

$$\langle m(\mathbf{x})m(\mathbf{x}+\mathbf{r})\rangle_{T_c} = \tfrac{1}{Kr^{d-2}}F(\tfrac{r}{a})$$

at the critical temperature T_c. Here, K is the so-called "stiffness constant" which appears in the (dimensionless) Ginzburg-Landau free-energy $\mathcal{F}[m] = \beta\mathcal{H}[m]$ via a gradient-square term $\int d^dx\, K|\boldsymbol{\nabla} m(\mathbf{x})|^2$ and "a" is a short-distance cut-off, e.g. a lattice-spacing in a crystalline lattice. The Landau (1935) mean-field theory prediction is obtained if one assumes that

$$\lim_{x\to\infty} F(x) = C < \infty$$

so that

$$\langle m(\mathbf{x})m(\mathbf{x}+\mathbf{r})\rangle_{T_c} \sim \tfrac{C}{Kr^{d-2}}, \quad r \gg a.$$

However, it is known that (for $d < 4$) instead that

$$F(x) \sim Cx^{-\eta}, \; x \gg 1$$

with $\eta > 0$, so that

$$\langle m(\mathbf{x})m(\mathbf{x}+\mathbf{r})\rangle_{T_c} \sim \tfrac{C}{Kr^{d-2}}(\tfrac{a}{r})^{\eta}, \; r \gg a.$$

In statistical physics, this is called anomalous scaling and η is called an anomalous dimension. Kenneth G. Wilson won the Nobel Prize in physics in 1982 for his renormalization group (RG) theory which provided an analytical framework to go beyond dimensional analysis and to derive anomalous dimensions like η. For example, see:

N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group (Westview Press, 1992) for a lucid account. Note that RG theory does not necessarily lead to power-laws, which correspond to fixed-points of the RG. More complex scaling behaviors may arise from other dynamical behaviors of the RG-flow (e.g. periodic oscillations in the length-scale r, corresponding to limit cycles of the RG). If anomalous scaling does occur, it coincides with what Barenblatt calls "incomplete similarity."

Returning to turbulent channel & pipe flow, the starting point of Bareblatt's considerations was another derivation of the log-law of the wall that was first given by Lev Landau in 1944:

> L. D. Landau & E. M. Lifshitz, Mekhanika Sploshnykh Sred (Gostekhizdat, Moscow, 1944) [1st edition of Fluid Mechanics, in Russian]

Landau, influenced by K41 theory, gave a derivation of the log-law by dimensional analysis considerations. Since by DA

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_*}{y}\Phi(y^+, Re_*),$$

one obtains

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_*}{y}\Phi(y^+),$$

if one assumes that $\Phi(y^+) = \lim_{Re_* \to \infty} \Phi(y^+, Re_*)$ exists. However, Landau then made the further assumption of complete similarity, so that also

$$\frac{1}{\kappa} = \lim_{y^+ \to \infty} \Phi(y^+) < +\infty$$

and thus

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_*}{\kappa y}, \ \ y^+ \gg 1.$$

Integration yields the usual log-law of the wall

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln y^+ + B, \ \ y^+ \gg 1.$$

Barenblatt's main conjecture is that the law of the wall for $y^+ \gg 1$ may instead correspond to a case of incomplete similarity, with

$$\Phi(y^+) \sim \alpha C (y^+)^\alpha \ \ \text{for} \ \ y^+ \gg 1$$

This result may be substituted into the formula for $\partial \bar{u}/\partial y$ and integrated, to yield:

$$\frac{\bar{u}}{u_*} \cong C(y^+)^\alpha \ \ \text{for} \ \ y^+ \gg 1.$$

Fitting to the pipe-flow data of

J. Nikuradze, "Gesetzmässigkeiten der turbulenten Strömung in glatten Röhren," VDI Forschungsheft No. 356 (1932),

Barenblatt obtained

$$\alpha = \frac{3}{2\ln Re}$$

$$C = \frac{1}{\sqrt{3}}\ln Re + \frac{5}{2}$$

with $Re = \frac{\bar{u}R}{\nu}$, where $\bar{u}$ is the bulk velocity in the pipe and R is the radius. It should be emphasized that these are empirical fits with no theoretical derivation. Likewise, the power-law with exponent α is an assumption, not derived. It is a bit unusual that Barenblatt's hypotheses imply that even the limit $Re \to \infty$ is not allowed! The analogous limit is "safe" in the other examples of incomplete similarity that we discussed above. On the other hand, no fundamental principle of physics is violated by these assumptions (so far as we know). Taking a fixed $y^+ \gg 1$ and letting $Re \to \infty$, the Barenblatt formula gives

$$\frac{\bar{u}}{u_*} \sim \frac{1}{\sqrt{3}}\ln Re + \frac{5}{2} + \frac{\sqrt{3}}{2}\ln y^+, \;\; \text{fixed } y^+ \gg 1 \;\; Re \to \infty$$

This formula violates a basic assumption of the log-layer theory, that $\bar{u}/u_*$ remains finite as $Re \to \infty$. However, one should remember that this is only an assumption and that alternatives may be considered.

A careful critique of the Barenblatt-Chorin theory is given by R. L. Panton (2005), Section 7.5. He observes that the Barenblatt formula is not a standard (Poincáre) asymptotic series. However, this cannot in our view be a sound criticism, because turbulent channel & pipe flow may require a more complex asymptotic description. Panton also criticizes the B-C theory because it does not provide global asymptotics, but applies only in an intermediate layer, unlike the traditional theory by asymptotic matching, where composite expansions give a description across the entire flow width/radius. Also, Panton claims that the Isakson-Millikan matching argument demonstrates "complete similarity."

This assertion is partly true but must be treated with great caution. The Isakson-Millikan argument is exact, given its basic assumptions. However, these assumptions may be incorrect! To emphasize this point, let us repeat the matching analysis under a slightly different set of assumptions. Recalling that

$$\ln x = \lim_{\alpha \to 0} \frac{x^{\alpha}-1}{\alpha},$$

Let us explore the consequence of assuming that the velocity profile in the inner scaling satisfies

$$\bar{u}/u_* \equiv f(y^+, Re_*) \sim \frac{(y^+)^{\alpha}-1}{\alpha\kappa} + B$$

and the friction law satisfies

$$\bar{u}_c/u_* \sim \frac{Re_*^{\alpha}-1}{\alpha\kappa} + C$$

for $Re_* \gg y^+ \gg 1$. If α is small, then this set of assumptions should be nearly the same as in the standard theory. These two relations are consistent with an outer scaling for $Re_* \gg 1, \eta \ll 1$:

$$\frac{\bar{u}-\bar{u}_c}{u_*} \equiv F(\eta, Re_*) \sim \frac{(y^+)^{\alpha}-Re_*^{\alpha}}{\alpha\kappa} + A = Re_*^{\alpha}\left(\frac{\eta^{\alpha}-1}{\alpha\kappa}\right), \quad A = B - C$$

using $y^+ = Re_*\eta$. It is easy to check that these expressions satisfy the matching relation

$$\eta\frac{dF}{d\eta} = \frac{Re_*^{\alpha}\eta^{\alpha}}{\kappa} = \frac{(y^+)^{\alpha}}{\kappa} = y^+\frac{df}{dy^+}$$

in the overlap region $\eta \ll 1, y^+ \gg 1$ for $Re_* \gg 1$. This is clearly an example of "incomplete similarity" since

$$y^+\frac{df}{dy^+} \not\longrightarrow \text{ constant}$$

for $Re_* \gg y^+ \gg 1$, but instead has power-law scaling y^+. Unfortunately, the power-law factor Re_*^{α} in the outer-law scaling relation is strongly Reynolds-number dependent, inconsistent with observations, if α is a fixed positive number. Thus, let us assume instead that

$$\alpha = \frac{\gamma}{\ln Re_*}$$

for a positive constant γ. In that case,

$$Re_*^{\alpha} = e^{\gamma}$$

becomes <u>independent</u> of Reynolds number and determined solely by γ. In terms of γ, we can then rewrite

$$\frac{\bar{u}}{u_*} \sim \frac{\ln Re_*}{\kappa\gamma}((y^+)^{\gamma/\ln Re_*} - 1) + B, \quad Re_* \gg y^+ \gg 1$$

$$\frac{\bar{u}_c}{u_*} \sim \frac{e^{\gamma}-1}{\kappa\gamma}\ln Re_* + C, \quad Re_* \gg 1$$

$$\frac{\bar{u}-\bar{u}_c}{u_*} \sim \frac{e^{\gamma}\ln Re_*}{\kappa\gamma}(\eta^{\gamma/\ln Re_*}-1)+A, \quad Re_* \gg 1, \eta \ll 1$$

It is remarkable that the friction law has the same form as in the log-layer theory, but with a modified von Kármán constant

$$\frac{\bar{u}_c}{u_*} \sim \frac{1}{\kappa_c}\ln Re_* + C, \quad \kappa_c \equiv \frac{\gamma\kappa}{e^{\gamma}-1}$$

for $Re_* \gg 1$. Similarly, we see that

$$\frac{u}{u_*} \sim \frac{1}{\kappa_i}\ln y^+ + B, \quad \kappa_i \equiv \kappa$$

for fixed $y^+ \gg 1$ and $Re_* \to \infty$, and

$$\frac{\bar{u}-\bar{u}_c}{u_*} \sim \frac{1}{\kappa_0}\ln \eta + A, \quad \kappa_0 \equiv e^{-\gamma}\kappa$$

for fixed $\eta \ll 1$ and $Re_* \to \infty$. Thus, the standard expressions of the log-layer theory are asymptotically valid, but with different values of the von Kármán constant for the logarithmic friction law, the logarithmic outer law and the logarithmic law of the wall!

This example illustrates vividly the conclusion that the Isakson-Millikan matching argument does not lead inevitably to the standard log-layer theory and that other essentially different scalings are logically possible and consistent with the RANS equations. A similar conclusion was reached by

> N. Afzal, "Power law and log law velocity profiles in fully developed turbulent pipe flow: equivalent relations at large Reynolds numbers," Acta Mechanica **151** 171-183 (2001)

Our result that different Kármán constants $(\kappa_c, \kappa_0, \kappa_i)$ are possible appears to be new. The above one-parameter family of scaling laws, indexed by $\gamma > 0$, all appear to be logically possible. The case $\gamma = 0$ is the standard theory of the log-layer, with all $\kappa_c = \kappa_0 = \kappa_i$. One can doubtless get better agreement with experiment by using the additional constant γ as a fitting parameter. Notice that uniformly valid composite expansions may be formed, if one assumes that Reynolds-number-independent limits exist for the wake function

$$W(\eta) = \lim_{Re_* \to \infty}[F(\eta, Re_*) - (\frac{e^{\gamma}\ln Re_*}{\kappa\gamma}(\eta^{\gamma/\ln Re_*}-1)+A)]$$

and for the viscous-buffer layer function

$$v(y^+) = \lim_{Re_*\to\infty}[f(y^+, Re_*) - (\frac{\ln Re_*}{\kappa\gamma}((y^+)^{\gamma/\ln Re_*} - 1) + B)]$$

so that

$$\frac{\bar{u}}{u_*} = v(y^+) + W(\eta) + [\frac{\ln Re_*}{\kappa\gamma}((y^+)^{\gamma/\ln Re_*} - 1) + B]$$

is uniformly valid across the channel /pipe. A similar composite expansion for the Reynolds stress is then

$$\frac{-\overline{u'v'}}{u_*^2} = 1 - \frac{dv}{dy^+} - \frac{(y^+)^{\gamma/\ln Re_* - 1}}{\kappa} - \eta.$$

See also Afzal (2001).

No claim of uniqueness is made in the above analysis. Many other asymptotic scaling results are doubtless possible and consistent with the RANS equations, but based on different fundamental hypotheses. Asymptotic matching arguments cannot settle which of these various scalings is correct. Careful experiments and simulations at very high Reynolds numbers will help to illuminate this issue. Better physical understanding is also required. For example, in the case of anomalous scaling of the structure functions, there is a clear physical intuition — based on "intermittency" and "multifractality" — that explains the origin of the scaling laws. We lack a similar physics-based explanation of the proposed corrections to the log-layer theory, e.g. in terms of fluctuations of momentum flux, vortex-structures, etc.