(E) High Reynolds-Number Scaling of Channel & Pipe Flow

We shall now discuss the scaling of statistical quantities in turbulent pipe and channel flow in the limit of high Reynolds numbers. Although an old subject, it is also a very active one at the present time, because of new developments in experiment, simulation and theory. Many of the key issues are still hotly debated and the subject is in a state of flux. For an excellent current review, we recommend

> "Scaling and structure in high Reynolds number wall-bounded flows," Theme Issue in Philos. Trans. Roy. Soc. A **365** (2007) 635-876

a collection of fourteen articles by leading researchers in the field. We shall discuss some of the findings and conclusions in those papers further below.

We shall first consider the scaling of the two primary quantities, the mean velocity $\bar{u}(y)$ and the Reynolds stress $\overline{u'v'}$, in turbulent channel flow. To begin, we shall follow closely the traditional treatment in Tenekes & Lumley, Section 5.2. For a more careful treatment along the same lines, using methods of asymptotic analysis, see:

> R.L. Panton, "Composite asymptotic expansions and scaling wall turbulence," Philos. Trans. Roy. Soc. A **365** 733-754 (2007)

Starting from the basic equation of the total stress $\tau_{xy}^{\rm tot}$

$$-\overline{u'v'} + \nu \frac{\partial \bar{u}}{\partial y} = u_*^2 (1 - \frac{y}{h}) \qquad (\star)$$

we see that $\overline{u'v'} \sim O(u_*^2)$ at high Reynolds number, if the viscous term can be neglected at finite distances from the wall. Also, the turbulent length-scale away from the walls must be set by the channel half-height h. This suggests the dimensionless outer scaling of variables, as

$$\eta = \frac{y}{h}, \; \frac{dF}{d\eta} = \frac{\partial \bar{u}/\partial y}{u_*/h}, \; G = \frac{-\overline{u'v'}}{u_*^2}$$

in terms of which $(\star)$ becomes

$$G + \frac{1}{Re_*} \frac{dF}{d\eta} = 1 - \eta \qquad (O)$$

with

$$Re_* = \frac{u_* h}{\nu}$$

the Reynolds number based on friction velocity u_* and half-height h. The dimensionless quantities F and G are some functions of η and Re_*:

$$\frac{dF}{d\eta} = \frac{dF}{d\eta}(\eta, Re_*),\ \ G = G(\eta, Re_*)$$

Taking $Re_* \gg 1$, the equation (O) suggests that

$$G(\eta, Re_*) \sim G_0(\eta) = 1 - \eta, \qquad Re_* \gg 1$$

as the leading-order asymptotics, with η fixed. Note, however, that

$$G(0, Re_*) = 0$$

because of the stick b.c. at the wall, whereas $G_0(0) = 1$! This is an indication that the high Re_*-asymptotics is a singular perturbation problem and that a layer with different physics exists near the wall. In particular, there will be influence from viscosity at distance from the wall of the order

$$d = \nu / u_*.$$

This motivates us to consider a different set of dimensionless variables corresponding to that layer at the wall, the inner scaling:

$$y^+ = \frac{y}{d} = \frac{y u_*}{\nu},\ \ f = \frac{\bar{u}}{u_*},\ \ g = \frac{-\overline{u'v'}}{u_*^2}$$

In these variables, the Reynolds stress equation becomes

$$g + \frac{df}{dy^+} = 1 - \frac{y^+}{Re_*} \qquad (I)$$

with

$$f = f(y^+, Re_*),\ \ g = g(y^+, Re_*),\ \ Re_* = h^+.$$

The equation (I) suggests that for $Re_* \gg 1$,

$$f(y^+, Re_*) \sim f_0(y^+), g(y^+, Re_*) \sim g_0(y^+), \qquad Re_* \gg 1$$

with

$$g_0 + \frac{df_0}{dy^+} = 1.$$

In the <u>inner layer</u> or <u>wall layer</u>, where this approximation is accurate, the stress is constant and does not vary perceptibly with the distance to the wall. The relations

$$\frac{\bar{u}(y)}{u_*} \cong f_0(y^+)$$

$$-\overline{u'v'}/u_*^2 \cong g_0(y^+)$$

are called the law of the wall. We shall discuss later the test of these relations and empirical results for the scaling functions f_0, g_0.

We have, so far, no information on the mean velocity $F(\eta, Re_*)$ in the outer layer or core region of the flow. For this purpose, we may consider the turbulent energy balance

$$-\overline{u'v'}\frac{\partial \bar{u}}{\partial y} = \varepsilon + \frac{\partial}{\partial y}[\overline{(p' + \frac{1}{2}q^2)v'} - \nu\frac{\partial}{\partial y}(\overline{\frac{1}{2}q^2})]$$

We can expect that

$$\varepsilon \sim O(\frac{u_*^3}{h}), \;\; \overline{(p' + \frac{1}{2}q^2)v'} \sim O(u_*^3), \;\; \frac{1}{2}\overline{q^2} \sim O(u_*^2)$$

which motivates us to define

$$\epsilon = \frac{\varepsilon}{u_*^3/h}, \;\; J = \frac{\overline{(p'+\frac{1}{2}q^2)v'}}{u_*^3}, \;\; K = \frac{\frac{1}{2}\overline{q^2}}{u_*^2}$$

so that, in the outer scaling,

$$G\frac{dF}{d\eta} = \epsilon + \frac{d}{d\eta}[J + \frac{1}{Re_*}\frac{dK}{d\eta}]$$

we therefore expect that for $Re_* \gg 1$

$$\frac{dF}{d\eta}(\eta, Re_*) \sim \frac{dF_0}{d\eta}(\eta)$$

in order to make both sides of order one. To get information on the mean velocity in the outer layer, this equation, written as

$$\frac{\partial \bar{u}}{\partial y} \cong \frac{u_*}{h}\frac{dF_0}{d\eta}$$

must be integrated from the centerline $\eta = 1$, to yield

$$\bar{u}(y) - \bar{u}_c = u_* F_0(\eta)$$

where $\bar{u}_c = \bar{u}(h)$ is the velocity at the center of the channel. This relation, usually written as

$$\frac{\bar{u}-\bar{u}_c}{u_*} = F_0(\eta),$$

is called the velocity-defect law in the outer layer.

At sufficiently large Re_*, there is a region where both outer layer and inner layer descriptions are simultaneously valid, with $\eta \to 0$ and $y^+ \to \infty$, respectively. Since

$$y^+/\eta = \frac{u_* h}{\nu} = Re_*$$

one may take, for example,

$$y^+ \sim Re_*^\alpha,\ \eta \sim Re_*^{\alpha-1}$$

with any $0 < \alpha < 1$. In such a range, $y^+ \to \infty, \eta \to 0$ as $Re_* \to \infty$. In this range, called the region of overlap or matched layer, one can match the two descriptions, giving

$$\frac{u_*}{h}\frac{dF_0}{d\eta} = \frac{\partial \bar{u}}{\partial y} = \frac{u_*^2}{\nu}\frac{df_0}{dy^+}$$

Multiplying this equation by y/u_* gives

$$\eta\frac{dF_0}{d\eta} = y^+\frac{df_0}{dy^+}.$$

If we consider finite (but large) y^+, then $\eta \cong 0$ as $Re_* \to \infty$. Likewise, if we consider finite (but small) η, then $y^+ \cong \infty$, $Re_* \to \infty$. Assuming that one of the limits $\lim_{\eta\to 0}\eta\frac{dF_0}{d\eta}$ or $\lim_{y^+\to\infty} y^+\frac{df_0}{dy^+}$ is a finite number, then the other must be the same number. We then see that the above expression must be nearly constant in the matching region:

$$\eta\frac{dF_0}{d\eta} = y^+\frac{df_0}{dy^+} \sim \frac{1}{\kappa}$$

These relations may be integrated to yield the logarithmic law of the wall:

$$F_0(\eta) \sim \frac{1}{\kappa}\ln\eta + A, \qquad \eta \ll 1$$

$$f_0(y^+) \sim \frac{1}{\kappa}\ln y^+ + B, \qquad y^+ \gg 1$$

with A and B constants. The constant κ is the so-called von Kármán constant. This logarithmic behavior was first suggested by

> T. von Kármán, "Mechanische Ähnlichkeit und Turbulenz," Nach. Ges. Wiss. Göttingen, Math.-Phys. Kl. 58-76 (1930)

and

> L. Prandtl, "Zur turbulenten Strömung in Röhren und längs Platten," Ergebn. Aerodyn. Versuchsanst, Göttingen **4** 18-29 (1932)

by different theoretical arguments. The above derivation using asymptotic matching is an (oversimplified) version of that given by

A. Isakson, "On the formula for the velocity distribution near the walls," Zh. Eksper. Teor. Fiz. **7** 919-924 (1937)

and

C. B. Millikan, "A critical discussion of turbulent flows in channels and circular tubes," Proc. 5th. Int. Congr. Appl. Mech. (Cambridge, MA 1939), pp. 386-392.

Because of the logarithmic variation of the mean velocity, the overlap region is often called the logarithmic layer. A corresponding result for the Reynolds stress can be obtained from the relation

$$g_0 + \frac{df_0}{dy^+} = 1,$$

which yields

$$-\frac{\overline{u'v'}}{u_*^2} = g_0(y^+) = 1 - \frac{1}{\kappa y^+}, \quad y^+ \gg 1$$

The logarithmic region is thus a range of approximately constant Reynolds stress and relatively small viscous stress $\sim O(1/y^+)$. For this reason, the overlap region is often called also the inertial sublayer.

Returning to the logarithmic defect law

$$\frac{\bar{u}-\bar{u}_c}{u_*} = \frac{1}{\kappa}\ln\eta + A, \quad \eta \ll 1$$

and the logarithmic law of the wall

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln y^+ + B$$

we see by subtraction that

$$\frac{\bar{u}_c}{u_*} = \frac{1}{\kappa}\ln Re_* + C$$

with $C = B - A$. This is called Prandtl's logarithmic friction law. It implies that if u_* (or $\partial\bar{p}/\partial x$) is held fixed as $Re_* \to \infty$, then $\bar{u}_c \to \infty$. Equivalently, if $\bar{u}_c$ is held fixed as $Re_* \to \infty$, then $u_* \to 0$ at a weak (logarithmic) rate. The above derivation has assumed that

$$\frac{dF}{d\eta} \equiv \frac{\partial\bar{u}/\partial y}{u_*/h} \to \frac{dF_0}{d\eta}(\eta) = O(1), \qquad Re_* \to \infty.$$

However, the friction law shows that it would not be consistent to assume that

$$\widetilde{F}(\eta) \equiv \frac{\bar{u}}{u_*} \to \tilde{F}_0(\eta) = O(1), \quad Re_* \to \infty$$

Instead $\bar{u}$ must be scaled by a logarithmically different velocity scale, e.g. $\bar{u}_c = \bar{u}(h)$. In that case,

$$\widetilde{F}(\eta, Re_*) \equiv \frac{\bar{u}(y)}{\bar{u}_c} = 1 + (\frac{u_*}{\bar{u}_c})F(\eta, Re_*).$$

See Panton (2007) and especially

R. L. Panton, "Review of wall turbulence as described by composite expansions," Appl. Mech. Rev. **58** 1-36 (2005)

for a comprehensive account. These papers review also the notion of "composite expansion" in the theory of matched asymptotics applied to wall-bounded turbulence. It is useful to say a word about this here. The logarithmic profile

$$F_0(\eta) \sim \frac{1}{\kappa} \ln \eta + A \equiv [F_0(\eta)]_{cp}, \quad \eta \ll 1$$

is called the common part of $F_0(\eta)$ since it matches onto the similar logarithmic behavior of $f_0(y^+) \sim \frac{1}{\kappa} \ln y^+ + B$ for $y^+ \gg 1$. One can thus define a composite expansion

$$\frac{\bar{u}(y)}{u_*} \cong f_0(y^+) + F_0(\eta) - [F_0(\eta)]_{cp}$$

which should be uniformly valid over the whole width of the channel. This is usually expressed in terms of the wake function.

$$W_0(\eta) = F_0(\eta) - [F_0(\eta)]_{cp} = F_0(\eta) - [\frac{1}{\kappa} \ln \eta + A]$$

which measures the deviation from the logarithmic profile. It satisfies $W_0(\eta) \to 0$ for $\eta \to 0$, but can be sizable for $\eta = O(1)$, near the centerplane or core of the flow. The composite expansion then becomes

$$\frac{\bar{u}(y)}{u_*} = f_0(y^+) + W_0(\eta), \ \eta = \frac{y^+}{Re_*}$$

which should be uniformly valid for all $0 < y < h$ as $Re_* \to \infty$.

Physical discussion: energetics

Let us consider some physical consequences of our asymptotic analysis. We have seen that in the inertial sublayer the Reynolds stress is approximately equal to u_*^2 and the mean velocity-gradient equal to $u_*/\kappa y$. Hence, the turbulence production is

$$-\overline{u'v'}\frac{\partial \bar{u}}{\partial y} \cong \frac{u_*^3}{\kappa y}$$

If turbulence production is mainly balanced by viscous dissipation, then (Taylor, 1935)

$$\varepsilon \cong \frac{u_*^3}{\kappa y}.$$

The picture is one of a Kolmogorov-type energy cascade in approximately homogeneous turbulence under weak local shear, with a spatial flux of energy transported across it from the outer flow to the wall. However, this spatial transport is expected to be a relatively small perturbation of the energy cascade in the inertial sublayer. For example,

> N. Marati, C. M. Casciola & R. Piva, "Energy cascade and spatial fluxes in wall turbulence," J. Fluid Mech. **521** 191-215 (2004)

have studied the issue in DNS. They find that energy that is produced at the shear length-scale

$$\ell_s = \sqrt{\frac{\varepsilon}{|\partial \bar{u}/\partial y|^3}}$$

is locally dissipated around the location y at the Kolomogrov microscale

$$\eta_K = (\frac{\nu^3}{\varepsilon})^{1/4} \cong (\frac{\kappa y \nu^3}{u_*^3})^{1/4} = \kappa^{1/4} y/y_+^{3/4}$$

Of course, this cannot be true arbitrarily close to the wall, because the integral length scale must be approximately

$$L \cong \kappa y,$$

since the largest eddies can have size, at most, the distance to the wall. This decreases with decreasing y even faster than does η_K. Thus,

$$\frac{L}{\eta_K} \cong (\kappa y^+)^{3/4} = (Re_y)^{3/4}$$

as y^+ decreases. The existence of a turbulent energy cascade requires that L/η_K be sufficiently large, above some critical value (roughly of order $10 - 100$). Thus, there is some region with distance $y^+ = O(1)$ from the wall, with no energy cascade and where viscous energy dissipation dominates. This is called the viscous sublayer. The velocity field in this range still fluctuates, since the local Reynolds number Re_y is transitional, but does not support an energy cascade. The mean velocity profile can be inferred in this range from

$$u_*^2 \cong -\overline{u'v'} + \nu \frac{\partial \bar{u}}{\partial y} \cong \nu \frac{\partial \bar{u}}{\partial y}$$

which implies that

$$\bar{u}/u_* \cong y^+$$

in the viscous sublayer. At somewhat larger distances y^+ neither one of the stresses can be neglected, in a region called the buffer layer. The turbulence production $-\overline{u'v'}\frac{\partial \bar{u}}{\partial y}$ reaches a maximum here. This may be seen by writing the production in dimensionless form

$$g_0 \frac{df_0}{dy^+}$$

and using

$$g_0 + \frac{df_0}{dy^+} = 1$$

which implies that the maximum value 1/4 is reached when Reynolds stress and viscous stress exactly balance, i.e. $g_0 = \frac{df_0}{dy^+} = \frac{1}{2}$. A schematic representation of the different ranges in channel flow is shown below:

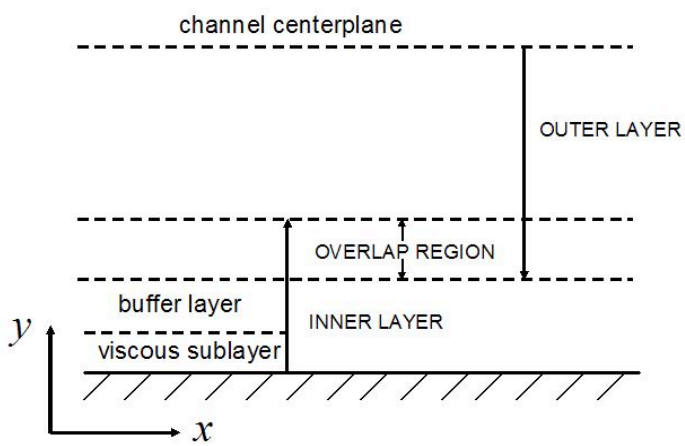

Figure 1.

The energy dissipation in the inertial sublayer (overlap region) $\varepsilon(y) \cong u_*^3/\kappa y$ also peaks at its inner limit, near the buffer layer. This leads to a logarithmic divergence in the total dissipation. If we integrate over the entire logarithmic layer, from a point near the outer edge $O(h)$ to a point near the inner edge $O(\nu/u_*)$, one obtains

$$\int_{\log-\text{layer}} \varepsilon(y)dy = u_*^3 \int_{O(\nu/u_*)}^{O(h)} \frac{dy}{\kappa y} \cong \frac{u_*^3}{\kappa} \ln(Re_*)$$

The source of the energy is mean-flow kinetic energy transferred into the wall layer by Reynolds stresses, i.e. through the spatial energy transport term

$$-\overline{u'v'} \cdot \bar{u}(y)$$

at the outer edge of the inertial sublayer. For $y = O(h)$ this gives

$$-\overline{u'v'} \cdot \bar{u}(y) \cong u_*^2 \bar{u}_c \cong \frac{u_*^3}{\kappa} \ln(Re_*),$$

matching the dissipation rate in the log-layer. Note that there is relatively little direct dissipation in the outer layer. This may be estimated by production, as

$$-\overline{u'v'} \cdot \frac{\partial \bar{u}(y)}{\partial y} = O(u_*^2 \frac{u_*}{h}) = O(\frac{u_*^3}{h})$$

which integrated across the height $O(h)$ of the core gives a net dissipation

$$O(u_*^3) \ll O(u_*^3 \ln(Re_*)).$$

Similarly, there is relatively little dissipation in the viscous sublayer. The local magnitude is very large

$$\nu(\frac{\partial \bar{u}}{\partial y})^2 = O(\frac{u_*^4}{\nu}),$$

but this is concentrated in a narrow layer of height $O(\nu/u_*)$. Thus, the net dissipation is again $O(u_*^3) \ll O(u_*^3 \ln Re_*)$. The final conclusion is that most of the energy dissipation occurs in the log-layer near the inner edge (buffer layer). This energy is provided by mean-flow energy transported to the walls by Reynolds stress and the mean-flow energy is, in turn, provided by pressure head.

It is interesting to consider Onsager's conjecture on inviscid energy dissipation for wall-bounded flows. If the channel flow is forced with $\bar{u}_c$ fixed, then, as we have seen earlier, the standard scaling theory implies that

$$u_* \sim \bar{u}_c / \ln Re_*, \qquad Re_* \gg 1. \tag{$*$}$$

Thus the energy dissipation $\varepsilon(y) \to 0$ pointwise in y as $Re_* \to \infty$ and also the energy dissipation integrated over the entire channel is predicted to vanish as $Re_* \to \infty$. Of course, the energy dissipation goes to zero only very weakly (logarithmically). It is not really surprising, because the mechanism of turbulence generation—the viscous boundary layer at the wall—is Reynolds-number dependent. Notice that if the pressure gradient $\partial\bar{p}/\partial x$ is held fixed instead, then energy dissipation becomes Re_*-independent (but in that case $\bar{u}_c \to \infty$ as $Re_* \to \infty$!) As we shall see in subsection (g), empirical evidence from simulations and experiments generally supports the scaling (*). We know of no direct study of local energy dissipation $\varepsilon(y)$ in channel flow or pipe flow, which investigates systematically its scaling with Re_*. In Chapter O, we mentioned the experimental work of Cadot et al. (1997) in Taylor-Couette cells with smooth walls, which found that energy dissipation in the boundary layer of such flows also decreases with Reynolds number but that the energy dissipation in the bulk appears to satisfy Taylor's relation $\varepsilon \sim U^3/L$ and to be independent of Reynolds number. It is therefore suggested that, at extremely high Reynolds numbers, the "inviscid" energy dissipation in the bulk may become greater than the dissipation in the near-wall log-layer and and buffer layer.

In this respect, there is an important mathematical result of Kato on wall-bounded flows:

> T. Kato, "Remarks on the zero viscosity limit for nonstationary Navier-Stokes flows with boundary," In: *Seminar on Nonlinear Partial Differential Equations*, S.S. Chern, eds. (Springer, NY, 1984).

since elaborated and extended by others, e.g.

> X. Wang, "A Kato type theorem on zero viscosity limit of Navier-Stokes flows," Indiana U. Math. J. **50** 223–241 (2001)

Assuming that a smooth Euler solution exists satisfying the no flow-through condition at the wall, Kato proved that the following two conditions are equivalent: (ii) integrated energy dissipation vanishes in a very tiny boundary layer of thickness $c\nu/u$ for a fixed constant c, and (ii) any Navier-Stokes solution with stick b.c. at the wall converges in strong L^2-sense to the Euler solution as $\nu \to 0$. The latter statement implies that energy dissipation must, in fact, vanish everywhere in the domain. Notice that the boundary layer in Kato's theorem goes to zero thickness even in wall units ν/u_*, if indeed $u_*/u \to 0$ as $\nu \to 0$. Thus, if energy dissipation vanishes very near the boundary—as present experiments suggest is true—then energy dissipation can remain in the bulk only if the Euler solution becomes singular in finite time.

vorticity transport

We have seen that energy dissipation/drag in wall-bounded flow requires a cross-stream flow of vorticity, and in channel flow (or pipe flow) a constant flow in the wall-normal direction

$$\overline{\Sigma}_{yz} = \overline{v'\omega_z' - w'\omega_y'} - \nu \tfrac{\partial \bar{\omega}_z}{\partial y} = -\sigma_*.$$

Because the vorticity flux is related in general to the divergence of the stress (minus the isotropic part), the vorticity flux in channel flow is related to the Reynolds and viscous stresses as:

$$\overline{\Sigma}_{yz} = -\tfrac{\partial}{\partial y}\tau_{xy}^{\rm tot}$$

with $\tau_{xy}^{\rm tot} = \overline{u'v'} - \nu\frac{\partial \bar{u}}{\partial y}$. Thus, our previous scaling analyses can be applied to the vorticity flux. The most important conclusion is that, in the near-wall region, the nonlinear advective transport contribution has the wrong (positive) sign. This can be inferred from

$$\overline{v'\omega_z' - w'\omega_y'} = -\tfrac{1}{d}\tfrac{\partial}{\partial y^+}\overline{u'v'} \sim +\tfrac{u_*^3}{\nu\kappa(y^+)^2} > 0.$$

Thus, the vorticity transport is dominated by viscous diffusion close to the wall.

How close? As the above relation makes clear, the advective vorticity transport changes sign at the maximum of the Reynolds stress in the wall normal direction. An extremum in the stress cannot be seen either in inner or in outer scaling separately, since in the former the stress is strictly increasing and in the latter strictly decreasing. To get a prediction for the location of the maximum stress, we must employ a uniformly valid expression. This has been done by Panton (2005), who used a composite expansion for Reynolds stress constructed from

$$g_0(y^+) \sim 1 - \frac{1}{\kappa y^+}, \quad G_0(\eta) \sim 1 - \eta, \quad [G_0(\eta)]_{\rm cp} = 1,$$

so that

$$-\overline{u'v'}/u_*^2 \simeq g_0(y^+) + G_0(\eta) - [G_0(\eta)]_{\rm cp} = 1 - \frac{1}{\kappa y^+} - \frac{y^+}{Re_*}.$$

An easy calculation (Panton, 2005) shows that the peak Reynolds stress occurs for

$$y_p^+ \sim (Re_*/\kappa)^{1/2}.$$

This result was already known empirically by data analyses of

> R. R. Long and T.-C. Chen, "Experimental evidence for the existence of the mesolayer in turbulent systems," J. Fluid Mech. **105** 19-59 (1981).
>
> K. R. Sreenivasan, "A unified view of the origin and morphology of the turbulent boundary layer structure," in *Turbulence Management and Relaminarisation*, edited by H. W. Liepmann and R. Narasimha (Springer- Verlag, Berlin, 1987), pp. 3761.

who used the results to argue for a "mesolayer" or "critical layer" in turbulent boundary layers. Here is the compilation of data from Long & Chen (1981):

Evidence for the existence of the 'mesolayer' 23

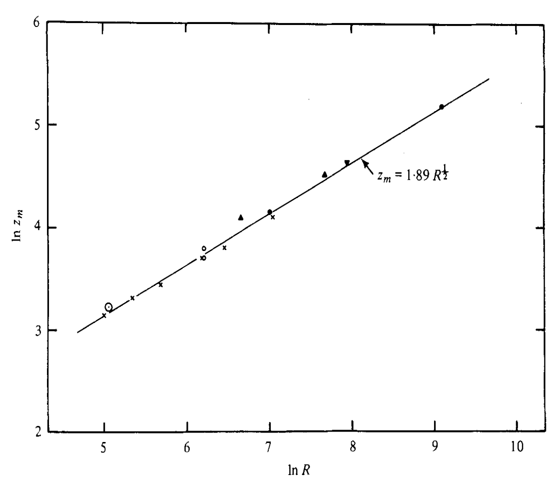

FIGURE 2. Distance z_m of maximum of Reynolds stress from wall for pipes and boundary layers. The line is $z_m = 1{\cdot}89R^{\frac{1}{2}}$, where $R = u_\tau a/\nu$ for a pipe and $u_\tau \delta_d/\nu$ for a boundary layer where δ_d is boundary-layer thickness. ×, Nikuradse, pipe; ●, Laufer, pipe; ⊙, Ueda & Mizushina, pipe; ▲, Gupta & Kaplan, boundary layer; ▼, Klebanoff, boundary layer; ○, Schildknecht *et al.*, boundary layer.

These results were originally quite controversial, because it was claimed that they invalidated the traditional scaling analyses leading to the log layer. The argument of Panton (2005) shows clearly that this is not the case.

However, there is a contradiction with some traditional physical interpretations of the log layer as a region where viscosity is negligible. Quoting Tennekes & Lumley (1971), p.156, "The matched layer is called *inertial sublayer* because of this absence of local viscous effects." As a matter of fact, the peak of the Reynolds stress occurs in the middle of the log-layer! (More precisely, intermediate length scales at high Re_* with $y^+ \sim Re_*^\alpha \to \infty$ and $\eta \sim Re_*^{\alpha-1} \to 0$ for any $0 < \alpha < 1$ define the matching region.) As we have seen, however, for $y^+ < y_p^+ \sim Re_*^{1/2}$ (and some distance beyond) viscous diffusion dominates in the vorticity transport.

effects of roughness

All of our discussion up until this point has assumed perfectly smooth walls. The surfaces of real pipes (and channels) will instead be rough at macroscopic scales:

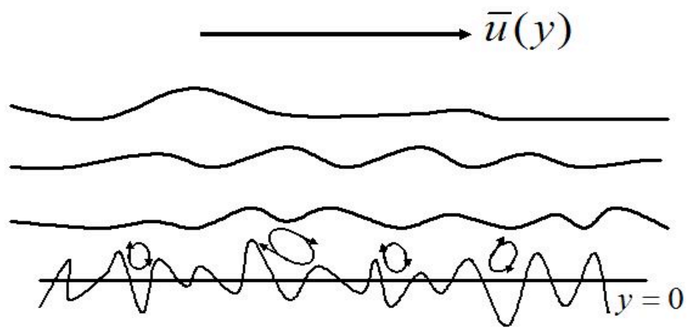

Figure 2.

The rms variation of the y-coordinate of the wall around the mean value $y = 0$ is called the roughness height k. This introduces a new fundamental length-scale into the problem, in addition to the friction length $d = \nu/u_*$ and the pipe radius R. Thus, dimensional analysis for the mean velocity gives (in inner scaling)

$$\frac{\bar{u}(y)}{u_*} = f(y^+, Re_k, Re_*)$$

or

$$\frac{\bar{u}(y)}{u_*} = \widetilde{f}(\tfrac{y}{k}, Re_k, Re_*)$$

where $Re_k = k^+ = ku_*/\nu$ is the roughness Reynolds number.

If f has a limit for $Re_* \to \infty$, then

$$\frac{\bar{u}(y)}{u_*} = f_0(y^+, Re_k)$$

or

$$\frac{\bar{u}(y)}{u_*} = \widetilde{f}_0(\tfrac{y}{k}, Re_k)$$

for pipe radius $R \gg y, k$. It is reasonable to expect that the velocity defect will be independent of Re_k for $k \ll R$:

$$\frac{\bar{u}(y) - \bar{u}_c}{u_*} = F(\ \eta, Re_*).$$

If we follow the traditional log-layer theory, then for $Re_* \to \infty$, $F(\eta, Re_*) \to F_0(\eta)$ so that

$$\frac{\bar{u}(y) - \bar{u}_c}{u_*} = F_0(\eta) \sim \tfrac{1}{\kappa} \ln \eta + A,\ \ \eta \ll 1.$$

This must be matched to the inner functions, which requires

$$\frac{\bar{u}(y)}{u_*} = \tfrac{1}{\kappa} \ln y^+ + B(Re_k),\ \ y^+ \gg 1$$

or

$$\frac{\bar{u}(y)}{u_*} = \tfrac{1}{\kappa} \ln(\tfrac{y}{k}) + \widetilde{B}(Re_k),\ \ y/k \gg 1$$

A similar analysis for the Reynolds stress gives

$$-\frac{\overline{u'v'}}{u_*^2} = g(y^+, Re_k, Re_*) = \widetilde{g}(y_k, Re_k, Re_*)$$

with $y_k \equiv y/k$. Scaling lengths with k,

$$\widetilde{g} + \frac{1}{Re_k}\frac{d\widetilde{f}}{dy_k} = 1 - (\tfrac{k}{R})y_k.$$

For $Re_* \to \infty$

$$\widetilde{g}_0(y_k, Re_k) + \frac{1}{Re_k}\frac{d\widetilde{f}_0}{dy_k}(y_k, Re_k) = 1$$

This equation implies that viscous stress is small at positions y_k of order 1 if $Re_k \to \infty$. Physically, the roughness elements at large Re_k produce turbulent wakes that generate substantial Reynolds stress on scale $y \sim O(k)$. It is only for $y \ll k$ that the viscous stresses come to dominate.

These considerations suggest that the limit $Re_k \to \infty$ is well-defined at $y_k \sim O(1)$ so that

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln y_k, + \widetilde{B}, \;\; y_k \gg 1, Re_k \gg 1$$

with $\widetilde{B} = \lim_{Re_k \to \infty} \widetilde{B}(Re_k)$. If one subtracts the above relation and the defect law, then one obtains a friction law in the presence of roughness

$$\frac{\bar{u}_c}{u_*} = \frac{1}{\kappa} \ln(\frac{R}{k}) + \widetilde{C}$$

with $\widetilde{C} = \widetilde{B} - A$. This result is independent of the Reynolds number Re_*! Note in this case that energy dissipation will also become independent of Reynolds number, asymptotically for $Re_* \gg 1$. This is in agreement with the findings of Cadot et al. (1997) for a Taylor-Couette flow with wall-riblets, who observed in that case that energy dissipation was insensible to the Reynolds number even in the near-wall boundary layer.

The above result should merge with Prandtl's logarithmic friction law as $k \to 0$. Based on empirical data available at the time

> C. F. Colebrook, "Turbulent flow in pipes, with particular reference to the transitional region between smooth and rough wall laws," J. Inst. Civil Eng. **11** 133-156 (1939)

developed a transitional curve of the form

$$\frac{\bar{u}_c}{u_*} = \widetilde{C} - \frac{1}{\kappa} \ln\left(\frac{k}{R} + \frac{C'}{Re_*}\right)$$

which interpolated between the two results, depending upon the relative magnitudes of R/k and Re_* (or, equivalently, of k and $d = \nu/u_*$. Colebrook's formula predicts a monotonic decrease of the friction coefficient $\lambda = 2(u_*/\bar{u}_m)^2$ toward the totally rough result as $Re_* \to \infty$ (or $Re \to \infty$). In fact, this disagrees with the experiment of

> J. Nikuradse, "Laws of flow in rough pipes," VDI Forschungsheft **361** (1933); English translation in NACA Tech. Memo. 1292 (1950)

or, for a presentation of the results,

Modern Developments in Fluid Dynamics, ed. S. Goldstein, (Clarendon Press, Oxford, 1938), vol. II, Section VIII, 167.

Nikuradse's measurements of λ show a dip below the completely rough results before eventually saturating at the rough value of λ as $Re \to \infty$. We shall discuss later the modern experimental confirmation of this effect. The results are still not understood very well theoretically, however. For a recent attempt, see

G. Gioia & P. Chakraborty, "Turbulent friction in rough pipes and the energy spectrum of the phenomenological theory," Phys. Rev. Lett. **96** 044502 (2006)