(D) RANS Equations of Pipe Flow

As another example, we consider turbulent flow down a circular pipe of constant radius R. This example is very similar to channel flow, so we may be briefer. The geometry is

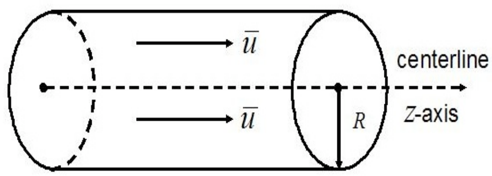

Figure D.1.

It is convenient to use cylindrical coordinates (r, θ, z), with the centerline of the pipe taken to be the z-axis. The mean flow is assumed to be in the z-direction, driven by a pressure-gradient in that direction. The various directions are called

radial or wall-normal $\quad r$-direction

azimuthal $\quad \theta$-direction

streamwise or axial $\quad z$-direction

Similarly as for channel flow, the velocity statistics may be assumed independent of θ, z and t, if the pipe is long enough and if the flow has been allowed to become statistically steady. Then all derivatives of averages with respect to these variables will be zero, except for $\partial\bar{p}/\partial z$, which maintains the flow against all stresses. All other averages, such as the mean flow velocity $\bar{u}_z(r)$, depend only upon the radial distance r. Its plot looks very similar to that for channel flow $\bar{u}(y)$ if the latter is plotted as a function of $y' = y - h$.

The Navier-Stokes equations in cylindrical coordinates may be found, for example, in G. K. Batchelor, An Introduction to Fluid Dynamics (Cambridge, 1967) Appendix 2. We need here the equations only for the z- and r-components of velocity:

$$\begin{aligned} \frac{\partial u_z}{\partial t} + \frac{\partial}{\partial z}(u_z^2) + \frac{1}{r}\frac{\partial}{\partial r}(r u_r u_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta u_z) &= -\frac{\partial p}{\partial z} + \nu \triangle u_z \\ \frac{\partial u_r}{\partial t} + \frac{\partial}{\partial z}(u_z u_r) + \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta u_r) + \frac{u_r^2 - u_\theta^2}{r} &= -\frac{\partial p}{\partial r} + \nu(\triangle u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}) \end{aligned}$$

where $\triangle$ is the Laplacian operator on scalar functions,

$$\triangle f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

With our statistical assumptions, these become

$$z\text{-component:} \quad \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'_r u'_z}) = -\frac{\partial \bar{p}}{\partial z} + \nu\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \bar{u}_z}{\partial r})$$

$$r\text{-component:} \quad \frac{\partial}{\partial r}(\overline{u_r'^2}) + \frac{1}{r}(\overline{u_r'^2 - u_\theta'^2}) = -\frac{\partial \bar{p}}{\partial r}$$

By differentiating the r-equation with respect to z, one finds that

$$\frac{\partial}{\partial r}(\frac{\partial \bar{p}}{\partial z}) = \frac{\partial}{\partial z}(\frac{\partial \bar{p}}{\partial r}) = 0$$

so that $\partial \bar{p}/\partial z$ is r-independent. Thus, one can multiply the z-equation by r and integrate, to obtain

$$r\overline{u'_r u'_z} = -\frac{1}{2}r^2\frac{\partial \bar{p}}{\partial z} + \nu r\frac{\partial \bar{u}_z}{\partial r}, \ \ 0 < r < R$$

Setting $r = R$ and using

$$\overline{u'_r u'_z} = 0, \ -\nu\frac{\partial \bar{u}_z}{\partial r} = u_*^2, \ \text{ at } \ r = R$$

gives

$$Ru_*^2 = -\frac{1}{2}R^2\frac{\partial \bar{p}}{\partial z}$$

which is just the net momentum balance in the z-direction

$$2\pi R u_*^2 = \pi R^2(-\frac{\partial \bar{p}}{\partial z}),$$

giving equality of stress integrated over the surface of the pipe and pressure-gradient integrated over the cross-sectional area. Alternatively, $-\partial \bar{p}/\partial z = 2u_*^2/R$. Substituting this result back into the equation for general r gives

$$\overline{u'_r u'_z} - \nu\frac{\partial \bar{u}_z}{\partial r} = u_*^2\frac{r}{R}$$

This is the same as the equation for channel flow if one identifies r with $y' = y - h$ there.

It is straightforward to consider also the energy balances, both of the mean velocity field and of the turbulent velocity fluctuations. The results are essentially identical to those for channel flow, so we leave that as an exercise. However, we shall consider here the vorticity dynamics, which has some geometric differences from the case of channel flow. The mean vorticity has only an azimuthal component

$$\bar{\omega}_\theta(r) = -\frac{\partial \bar{u}_z}{\partial r}(r) > 0$$

The lines of the mean vorticity thus correspond to vortex rings. From the equation for the θ-component of vorticity

$$\begin{aligned} \frac{\partial \omega_\theta}{\partial t} &= \frac{\partial}{\partial z}\left[u_\theta \omega_z - u_z \omega_\theta - \nu\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right)\right] \\ &\quad - \frac{\partial}{\partial r}\left[u_r \omega_\theta - u_\theta \omega_r - \nu\left(\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\theta) - \frac{1}{r}\frac{\partial \omega_r}{\partial \theta}\right)\right] \end{aligned} \tag{5}$$

one gets that

$$\frac{\partial}{\partial r}\overline{\Sigma}_{r\theta} = 0$$

with

$$\overline{\Sigma}_{r\theta} = \overline{u_r'\omega_\theta' - u_\theta'\omega_r'} - \frac{\nu}{r}\frac{\partial}{\partial r}(r\bar{\omega}_\theta).$$

Thus, the flux of azimuthal vorticity in the radial direction is constant (independent of r), so that one may consider turbulent pipe flow as sustaining simultaneously "spatial cascades" of momentum and vorticity in the radial direction.

The z-component of the momentum equation may be further written as

$$\frac{\partial u_z}{\partial t} = u_r\omega_\theta - u_\theta\omega_r - \nu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\theta) - \frac{1}{r}\frac{\partial \omega_r}{\partial \theta}\right] - \frac{\partial}{\partial z}\left(p + \frac{1}{2}|\mathbf{u}|^2\right)$$

so averaging gives

$$\overline{\Sigma}_{r\theta} = \frac{\partial \bar{p}}{\partial z} < 0$$

The picture is as follows:

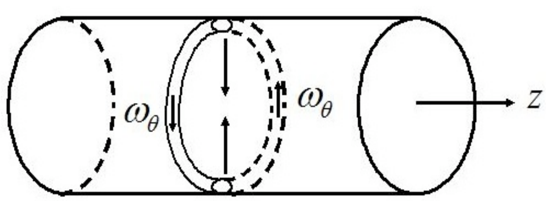

Figure D.2.

Rings of azimuthal vorticity form at the pipe wall, generated by the axial pressure gradient $\partial\bar{p}/\partial z$. These rings then shrink toward the centerline, where they annihilate at zero radius. Of course, this is only an accurate description "in the mean" and the true vortex dynamics is more complex. Cf. the earlier discussion for channel flow.