(B) Reynolds Averaging

Throughout most of our discussion, we have used as an averaging procedure the "coarse-graining" or "filtering approach." As we shall discuss in more detail below, this still applies in the presence of walls. However, it is also useful here to consider a somewhat different and cruder averaging procedure usually known as "Reynolds averaging". If a flow is statistically stationary, then, for any quantity $F[\mathbf{u}(t)]$ depending upon the velocity $\mathbf{u}(t)$ at time t, we may define

$$\overline{F[\mathbf{u}]} = \lim_{T\to\infty} \frac{1}{T}\int_0^T F[\mathbf{u}(t)]dt$$

by a <u>long-time average</u>. In this section we shall always carefully distinguish between $\overline{F[\mathbf{u}]}$ as defined above and $\overline{(F[\mathbf{u}])_\ell}$ defined by coarse-graining at length-scale ℓ. If the turbulence is <u>not</u> statistically stationary, then the above definition is not very useful (and perhaps not even mathematically meaningful). Reynolds (1895) proposed in that case to consider

$$\overline{F[\mathbf{u}(t)]} = \frac{1}{2\tau}\int_{t-\tau}^{t+\tau} F[\mathbf{u}(t')]dt'$$

where it is assumed that $\overline{F[\mathbf{u}(t)]}$ varies only slowly on a very long time-scale T and τ is some "intermediate time-scale"

$$\tau_c \ll \tau \ll T,$$

where τ_c is a temporal correlation length of the quantity $F[\mathbf{u}(t)]$. This definition makes sense only under rather restrictive circumstances. Thus, it is often preferred to define $\overline{F[\mathbf{u}(t)]}$ by an <u>ensemble-average</u>

$$\overline{F[\mathbf{u}(t)]} = \int P_t[d\mathbf{u}]F[\mathbf{u}]$$

where P_t is a probability measure on $V \subset H^1(\Omega, \mathbb{R}^3)$ obtained by evolving an initial measure P_0 under the Navier-Stokes dynamics. Such "probabilistic Navier-Stokes solutions" may be shown to exist:

> M. I. Vishik & A. V. Fursikov, "Mathematical Problems of Statistical Hydrodynamics," (Springer-Verlag, 1988)

If the initial measure P_0 is a <u>stationary</u> statistical solution P_*, then $P_t = P_*$ for all $t \geq 0$ and thus $\overline{F[\mathbf{u}(t)]}$ is likewise independent. In fact,

$$\overline{F[\mathbf{u}]} = \int P_*[d\mathbf{u}]F[\mathbf{u}] = \lim_{T\to\infty} \frac{1}{T}\int_0^T F[\mathbf{u}(t)]dt$$

in this situation. For more details, see the book of Vishik & Fursikov above. Unfortunately, when the turbulence is not statistically stationary (or homogeneous in at least one direction), probabilistic expectation values can be a bit unphysical because no observations, in practice, ever correspond to averages over multiple realizations of the same experimental set-up! It would be more natural in such cases to characterize the entire statistical distribution.

Let us now consider the Reynolds-averaged Navier-Stokes (RANS) equations. If we define

$$\bar{\mathbf{u}}(\mathbf{x},t) = \overline{\mathbf{u}(\mathbf{x},t)}$$

$$\mathbf{u}'(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \bar{\mathbf{u}}(\mathbf{x},t)$$

then it is easy to derive

$$\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\boldsymbol{\nabla})\bar{\mathbf{u}} + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} = -\boldsymbol{\nabla}\bar{p} + \nu\triangle\bar{\mathbf{u}}, \;\; \boldsymbol{\nabla}\cdot\bar{\mathbf{u}} = 0$$

where

$$\tau_{ij} = \overline{u_i'u_j'}$$

is the Reynolds stress. This looks just like the equation that we derived earlier using the filtering approach, except that the sense of averaging is different. This equation is, of course, the balance equation for mean momentum (per mass) and can be rewritten as

$$\partial_t\bar{\mathbf{u}} + \boldsymbol{\nabla}\cdot[\bar{\mathbf{u}}\bar{\mathbf{u}} + p\mathbf{I} + \boldsymbol{\tau}^{\rm tot}] = 0$$

with

$$\boldsymbol{\tau}^{\rm tot} = \tau - 2\nu\bar{\mathbf{S}}$$

the total stress, both the Reynolds stress and the mean Newtonian stress.

We may consider all of the same balance equations for Reynolds-averaged quantities that we considered earlier for the filtering approach:

balance equation for energy (per mass) in the mean field:

$$\partial_t(\tfrac{1}{2}|\bar{\mathbf{u}}|^2) + \boldsymbol{\nabla}\cdot[(\tfrac{1}{2}|\bar{\mathbf{u}}|^2 + \bar{p})\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\boldsymbol{\tau} - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\bar{\mathbf{u}}|^2)] = \overline{\mathbf{S}} : \boldsymbol{\tau} - \nu|\boldsymbol{\nabla}\bar{\mathbf{u}}|^2$$

This looks just like the equation in the filtering approach. In this context, the quantity

$$-\overline{\mathbf{S}} : \boldsymbol{\tau}$$

is usually called the <u>turbulent energy production</u>, since it presents a loss term for the energy of the mean field and (as we shall see) a gain term for the energy of the fluctuation field $\mathbf{u}'$.

<u>balance equation for the Reynolds stress</u>:

$$\begin{aligned}\partial_t \tau_{ij} + \partial_k [\bar{u}_k \tau_{ij} + \tau_{ijk} + \delta_{ki} \overline{p' u'_j} + \delta_{kj} \overline{p' u'_i} - \nu \partial_k \tau_{ij}] \\ = -[\tau_{ik} \bar{u}_{j,k} + \tau_{j,k} \bar{u}_{i,k}] + \overline{p' S'_{ij}} - 2\nu \overline{u'_{i,k} u'_{j,k}} \end{aligned} \tag{3}$$

with

$$\tau_{ijk} = \overline{u'_i u'_j u'_k}$$

This, again, looks like the equation that was derived for subscale stress in the filtering approach and the various terms have a similar interpretation as was discussed there. One-half of the trace of this equation yields a <u>balance equation for the turbulent kinetic energy</u>

$$\tfrac{1}{2} q^2 = \tfrac{1}{2} |\mathbf{u}'|^2, \quad k = \tfrac{1}{2} \overline{q^2}$$

as

$$\partial_t k + \partial_k [\bar{u}_k k + \tfrac{1}{2} \overline{q^2 u'_k} + \overline{p' u'_k} - \nu \partial_k k] = -\tau_{ij} \overline{S}_{ij} - \nu \overline{u'_{i,j} u'_{i,j}}$$

We see that the "turbulence production" $-\tau_{ij}\overline{S}_{ij}$ enters this balance as a source term, so that the energy which is removed from the mean field feeds the turbulent fluctuations. The quantity

$$\epsilon = \nu \overline{u'_{i,j} u'_{i,j}} \geq 0$$

is usually called the <u>turbulent energy dissipation</u>.

Finally, we may derive also a <u>balance equation for the mean vorticity</u>, which may be written in various forms. For example, we get easily

$$\partial_t \bar{\omega}_i + \partial_j \overline{\Sigma}_{ji} = 0$$

with

$$\overline{\Sigma}_{ij} = \bar{u}_i \bar{\omega}_j - \bar{u}_j \bar{\omega}_i + \overline{u'_i \omega'_j - u'_j \omega'_i} + \nu \left(\frac{\partial \bar{\omega}_i}{\partial x_j} - \frac{\partial \bar{\omega}_j}{\partial x_i}\right).$$

Alternatively, one may write

$$(\partial_t + \bar{\mathbf{u}} \cdot \boldsymbol{\nabla}) \bar{\boldsymbol{\omega}} = (\bar{\boldsymbol{\omega}} \cdot \boldsymbol{\nabla}) \bar{\mathbf{u}} + \boldsymbol{\nabla} \times \mathbf{f}^R + \nu \triangle \bar{\boldsymbol{\omega}}$$

where

$$f_i^R = \partial_j \tau_{ij}$$

is the turbulent Reynolds force. Here one may substitute for $\mathbf{f}^R$ instead

$$\mathbf{f}^v = \overline{\mathbf{u}' \times \boldsymbol{\omega}'}$$

the turbulent vortex force. Finally, one may also write

$$\partial_t \bar{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times (\bar{\mathbf{u}} \times \bar{\boldsymbol{\omega}} + \mathbf{f}^v - \nu \boldsymbol{\nabla} \times \bar{\boldsymbol{\omega}})$$

From the above equations one may derive a balance equation for the enstrophy in the mean field $\frac{1}{2}|\bar{\boldsymbol{\omega}}|^2$. Using similar methods as those before, one can derive also an equation for the mean turbulent enstrophy $\zeta = \frac{1}{2}\overline{|\boldsymbol{\omega}'|^2}$. There will be left as exercises.

It is worthwhile to make here a few remarks on the applicability of the filtering approach in the presence of walls. In all of our discussion in the course, we have used filtering just as an analysis tool. For this purpose, there is little or no problem in the application to wall-bounded turbulence. If one considers a filter-kernel $G(\rho)$ which is compactly supported in a (dimensionless) ball of radius R, then

$$\bar{\mathbf{u}}_\ell(\mathbf{x}) = \int d^d r \; G_\ell(\mathbf{r}) \mathbf{u}(\mathbf{x} + \mathbf{r})$$

has all of its contribution from the interior of the flow domain Ω for any point $\mathbf{x} \in \Omega$ such that

$$\mathrm{dist}(\mathbf{x}, \partial\Omega) > R\ell.$$

Thus, filtered equations such as

$$\partial_t \bar{\mathbf{u}}_\ell(\mathbf{x}, t) + (\bar{\mathbf{u}}_\ell(\mathbf{x}, t) \cdot \boldsymbol{\nabla})\bar{\mathbf{u}}_\ell(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell(\mathbf{x}, t) = -\boldsymbol{\nabla} \bar{p}_\ell(\mathbf{x}, t) + \nu \triangle \bar{\mathbf{u}}_\ell(\mathbf{x}, t)$$

are still valid and useful, as long as $\mathrm{dist}(\mathbf{x}, \partial\Omega) > R\ell$. Of course, as $\mathbf{x} \to \partial\Omega$, one is forced to take $\ell \to 0$ as well. It turns out that this is physically satisfactory, because — as we shall see — the integral length L turns out to be proportional to the distance to the wall. Since we are interested in using the above equations to analyze cascade dynamics at length-scales $\ell < L$, there is no essential restriction of applicability imposed by the condition that $\ell < \frac{1}{R}\mathrm{dist}(\mathbf{x}, \partial\Omega) \cong L$.

There is some difficulty, however, if one wants to use the above equations not just for local analysis but instead for global analysis. For example, one cannot integrate the above equations over the entire domain Ω in order to study the balance of total momentum in the flow at length-scale ℓ. The integral then must exclude a "boundary layer" of thickness $O(\ell)$ at the surface:

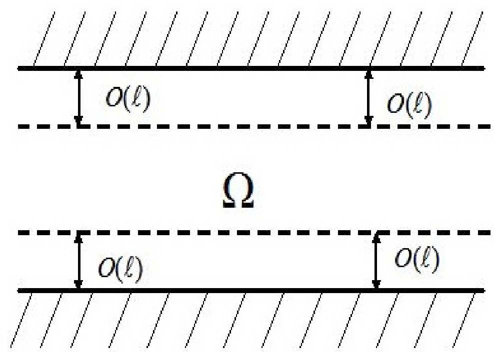

Figure B.1.

There will be fluxes of momentum across the surface of the "boundary layer" — the dotted lines in the figure above — that must then be understood.

There are related difficulties in large-eddy simulation (LES) modeling of wall-bounded turbulence. One cannot write globally valid filtered NS equations in Ω, which is a prerequisite to numerical solution! It is typical to use filter functions

$$\bar{\mathbf{u}}(\mathbf{x}) = \int d^d r \; G(\mathbf{r};\mathbf{x})\,\mathbf{u}(\mathbf{x}+\mathbf{r})$$

that depend upon $\mathbf{x}$, e.g. that have $\ell(\mathbf{x}) \to 0$ as $\mathrm{dist}(\mathbf{x},\partial\Omega) \to 0$. Such filtering does not, however, commute with space-gradients and this creates further analytical difficulties — so-called "commutation errors" — that complicate LES modeling. The decreasing filtering length-scale at the wall also makes the LES modeling method prohibitively expensive, since very fine grids must be employed near the wall. For a discussion of these issues, see e.g.

S. Chen et al. "Reynolds-stress-constrained large-eddy simulation of wall-bounded turbulent flows," J. Fluid Mech. **703** 1–28 (2012)