VI Wall-Bounded Turbulence

(A) The Navier-Stokes Equation in Bounded Domains

Before we can discuss turbulence in the presence of walls, we must first review some background material on the solution of the Navier-Stokes equations in bounded domains. For an excellent general discussion, we recommend

R. Teman, *Navier-Stokes Equations: Theory and Numerical Analysis* (North-Holland, Amsterdam, 1984)

This subject might seem to be rather cut and dried but there are — amazingly — still many confusions and controversies being discussed in the literature! There seems to be a great deal of misunderstanding, in particular, of the proper boundary conditions for pressure and vorticity. For example, the recent paper

D. Rempfer, "On the boundary conditions for incompressible Navier-Stokes problems," Applied Mechanics Reviews **59** 107-125 (2006)

extensively reviews the literature and points out many papers that still debate the "correct" b.c. for pressure and vorticity! In fact, this should not be a matter of controversy, as there are rigorous mathematical resolutions of those issues in the literature. There are open issues about certain numerical schemes for solving the incompressible Navier-Stokes equation, which we shall discuss briefly, but for the continuum equations the issues seem to be fully resolved.

Let us first discuss the appropriate boundary condition on the velocity $\mathbf{u}$. It may seem "obvious" that this should be the stick boundary condition (or no-slip b.c.)

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}$$

on a motionless boundary $\partial\Omega$, or, more generally,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_B(\mathbf{x}, t), \ (\mathbf{x}, t) \in \partial\Omega(t)$$

if the boundary is moving with velocity $\mathbf{u}_B(\mathbf{x},t)$ at point $(\mathbf{x},t)$. As a matter of fact, this issue was long debated in the scientific community, going back to D. Bernouli and L. Euler in the 18th century. For an interesting historical discussion, see:

> "Note on the Conditions at the Surface of Contact of a Fluid with a Solid Body," in Modern Developments in Fluid Dynamics, composed by the Fluid Motion Panel of the Aeronautical Research Committee and Others, ed. S. Goldstein (Clarendon Press, Oxford, 1938), reprinted by Dover, 1965; vol. II, pp. 676-680.

Many workers in the 18th and 19th century believed that there was a slip at the wall, so that the fluid velocity at the wall and the velocity of the wall itself would disagree, perhaps substantially. Navier proposed on the basis of a molecular argument that the slip velocity at the wall, $\delta\mathbf{u}_s$, should satisfy

$$\delta\mathbf{u}_s = \ell_s \frac{\partial \mathbf{u}}{\partial n} \qquad (*)$$

for an appropriate length-scale ℓ_s. Experiments carried out in the 19th century by Stokes, Whetham, Couette and others, however, showed no evidence of slip. Maxwell in 1879 used kinetic theory to argue that for a solid-gas interface

$$\ell_s \cong \ell_{mf}$$

where ℓ_{mf} is the mean-free length and thus $|\delta\mathbf{u}_s| \ll |\mathbf{u}|$ (except possibly in a very rare gas, or Knudsen gas). We now know that Maxwell's conclusion is correct and that for macroscopic fluid systems, slips at boundaries are generally very small. This is verified not only by many experiments, but also by much recent theoretical and numerical work. For gases see

> S. Shen et al., "A kinetic-theory based first-order slip boundary condition for gas flow," Phys. Fluids **19** 086101 (2007)

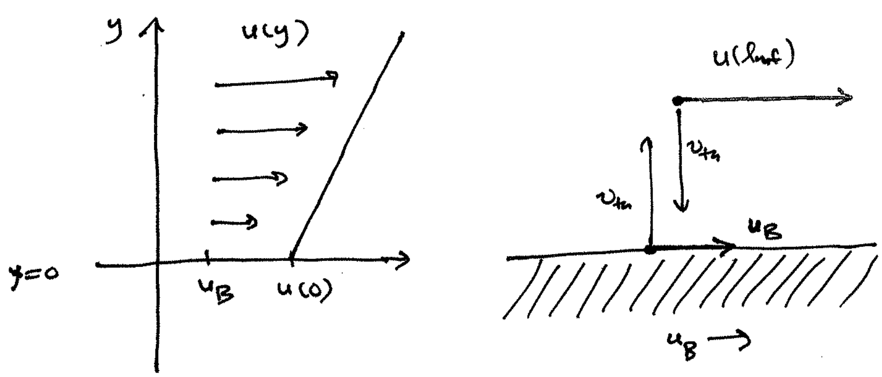
and for more general fluids

> C. Denniston and M. O. Robbins, "General continuum boundary conditions for miscible binary fluids from molecular dynamics simulations," J. Chem. Phys. **125**

214102 (2006)

In the case of simple liquids, Navier's law is a good approximation with $\ell_s \cong \ell_c$, the correlation length in the liquid. Thus, for macroscopic fluid flows, we see that slip velocities are generally quite small and can be set $= 0$.

It is interesting to give a simple kinetic-theory explanation for the origin of Navier's slip law $(*)$, following Navier and Maxwell. We thus consider a simple shear flow parallel to a solid wall moving with velocity u_B:



Then

$$
\begin{aligned}
T^{(1)}_{xy}(0) &= \text{ net flux of x-momentum in the y-direction at the wall } (y=0) \\
&= \rho u_B v_{th} - \rho u(\ell_{mf}) v_{th} \qquad (1)
\end{aligned}
$$

if we assume a very thin layer of gas molecules (much thinner than macroscopic scale) in thermal equilibrium with the molecules of the solid wall and having the same mean velocity u_B. Taylor expansion gives

$$u(\ell_{mf}) \cong u(0) + \ell_{mf} \tfrac{\partial u}{\partial y}(0).$$

so that, with $\delta u_s = u(0) - u_B$

$$T^{(1)}_{xy}(0) \cong -\rho v_{th} \delta u_s - \rho v_{th} \ell_{mf} \tfrac{\partial u}{\partial y}(0).$$

Now, at a point in the interior of the gas, with $y > 0$, the standard kinetic-theory argument gives

$$
\begin{aligned}
T_{xy}^{(1)}(y) &= \rho u(y-\ell_{mf})v_{th} - \rho u(y+\ell_{mf})v_{th} \\
&\cong -2\rho\ell_{mf}v_{th}\frac{\partial u}{\partial y}(y) \\
&= -\eta\frac{\partial u}{\partial y}(y) \qquad (2)
\end{aligned}
$$

with shear viscosity $\eta \cong -2\rho\ell_{mf}v_{th}$.

If we now consider the force balance on a small control volume of gas at the surface of the wall: then we can argue that, in steady state,

$$T_{xy}^{(1)}(0) \cong T_{xy}^{(1)}(y), \quad y \text{ macroscopically small } (y \cong 0)$$

so that

$$-\rho v_{th}\delta u_s - \rho v_{th}\ell_{mf}\tfrac{\partial u}{\partial y}(0) \cong -2\rho v_{th}\ell_{mf}\tfrac{\partial u}{\partial y}(0)$$

or

$$\delta u_s \cong \ell_{mf}\tfrac{\partial u}{\partial y}(0)$$

which is Navier's law, with $\ell_s \cong \ell_{mf}$. Note that this law was more traditionally written as

$$\beta\delta u_s = \eta\tfrac{\partial u}{\partial y}(0)$$

with $\beta \cong 2\rho v_{th}$ the so-called slip coefficient. Thus, $\eta/\beta = \ell_s$ defines the slip length.

The final conclusion is that, since $\ell_{mf} \ll L_\Delta = |\mathbf{u}|/|\boldsymbol{\nabla}\mathbf{u}|$ in macroscopic fluid flows — including, as we have seen, turbulent flows — the slip velocity δu_s may be ignored for most practical purposes. This is also shown by a great abundance of experimental evidence on turbulent flows past walls, which are consistent with zero (or very small) slips.

The problem thus becomes, mathematically, the following initial-boundary-value problem for the incompressible Navier-Stokes in an open Lipschitz domain Ω: Find a vector function

$$\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}$$

and a scalar function

$$p : \Omega \times [0,T] \to \mathbb{R}$$

such that

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}p - \nu\triangle\mathbf{u} = \mathbf{f}^B \text{ in } \Omega \times [0,T]$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \text{ in } \Omega\times[0,T]$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega$$

$$\mathbf{u}(\mathbf{x},t) = 0 \text{ in } \partial\Omega\times[0,T]$$

The body-force $\mathbf{f}^B$ and initial condition $\mathbf{u}_0$ are given as functions on $\Omega\times[0,T]$ and on Ω, respectively. We consider here the problem with stationary boundaries and stick velocities ($\mathbf{u}=\mathbf{0}$) in the boundary $\partial\Omega$. The problem with moving boundaries and non-zero velocities on the boundary can be handled by similar methods. For example, see

R. Salvi, "The exterior non-stationary problem for the Navier-Stokes equations in regions with moving boundaries," J. Math. Soc. Jap. **42** 495-509 (1990)

The standard mathematical theory of Leray weak solutions of the problem with stationary boundaries and stick conditions is in terms of the Hilbert space

$$H = \{\mathbf{u}\in L^2(\Omega,\mathbb{R}^d), \boldsymbol{\nabla}\cdot\mathbf{u}=0, \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0\}$$

with $\mathbf{n}$ the outward-pointing unit normal on $\partial\Omega$. This set of fields has finite energy, are incompressible (in distribution sense) and satisfy the condition of no-flow through the boundary. Another important space, of greater smoothness, is

$$V = \{\mathbf{u}\in H^1(\Omega,\mathbb{R}^d), \boldsymbol{\nabla}\cdot\mathbf{u}=0, \mathbf{u}|_{\partial\Omega}=\mathbf{0}\}$$

which satisfy also $\int_\Omega |\boldsymbol{\nabla}\mathbf{u}|^2\, d^dx < +\infty$, incompressibility, and the full stick conditions $\mathbf{u}=\mathbf{0}$ at the boundary $\partial\Omega$. The dual space

$$V' \subseteq H^{-1}(\Omega,\mathbb{R}^d)$$

is a space of divergence-free distributions that also plays a role in the Leray theory.

By projecting the Navier-Stokes equation onto the space H, one obtains the Leray weak form:

$$\frac{d\mathbf{u}}{dt} + \mathbf{B}(\mathbf{u},\mathbf{u}) + \nu A\mathbf{u} = \mathbf{f}^B \qquad (*)$$

$$A\mathbf{u} \equiv -P\triangle\mathbf{u}$$

$$\mathbf{B}(\mathbf{u},\mathbf{u}) = P((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u})$$

where P is the Helmholtz-Leray projection from $L^2(\Omega,\mathbb{R}^d)$ onto H, given by a Helmholtz decomposition theorem (see e.g. Theorem I. 1.5 in Temam, 1984). For simplicity we have assumed that $\mathbf{f}^B = P\mathbf{f}^B$, that is, that $\boldsymbol{\nabla} \cdot \mathbf{f}^B = 0$ and that $\mathbf{f}^B \cdot \mathbf{n} = 0$ on $\partial\Omega$. It is known that

$$A : V \to V'$$

and, for dimension $d \leq 4$,

$$B : V \times V \to V'.$$

Thus, the equation $(*)$ makes sense in the space V', interpreted weakly, that is, we seek

$$\mathbf{u} \in L^2(0,T;V),\ \tfrac{d\mathbf{u}}{dt} \in L^1(0,T;V')$$

such that

$$\langle \tfrac{d\mathbf{u}}{dt} + \mathbf{B}(\mathbf{u},\mathbf{u}) + \nu A\mathbf{u} - \mathbf{f}^B, \boldsymbol{\nu} \rangle = \mathbf{0}$$

for all $\boldsymbol{\nu} \in V$, with also

$$\mathbf{u}(0) = \mathbf{u}_0.$$

Here it is assumed that $\mathbf{f}^B \in L^2(0,T;V')$ and $\mathbf{u}_0 \in H$.

The <u>main theorem</u> of the subject is that at least one solution exists to the above problem, with, furthermore,

$$\mathbf{u} \in L^\infty(0,T;H)$$

and $\mathbf{u}$ weakly continuous from $[0,T]$ into H, i.e. $t \mapsto (\mathbf{u}(t), \boldsymbol{\nu})$ is a continuous function for all $\boldsymbol{\nu} \in H$. For example, see Theorem III. 3.1 of Temam (1984). It is noteworthy here that the initial conditions $\mathbf{u}_0$ need not satisfy the full stick boundary conditions, but only the condition of no-flow through the boundary. Nevertheless, at any <u>positive</u> time $t > 0$, the solution $\mathbf{u}(t) \in V$ and thus satisfies the stick conditions. This shows that, in general, there will be a <u>discontinuity</u> in the solution $\mathbf{u}(t)$ at the initial instant $t = 0$ at the boundary $\partial\Omega$, with

$$\mathbf{n} \times \mathbf{u} \neq \mathbf{0} \ \text{ at } t = 0 \text{ on } \partial\Omega$$

and

$$\mathbf{u} = \mathbf{0} \text{ at } \ t > 0 \text{ on } \partial\Omega.$$

It is only for <u>specially-prepared initial data</u> that the Navier-Stokes solution can be smooth up to $t = 0$. For a full discussion of these issues, we refer to

R. Temam, "Behavior at time $t = 0$ of the solutions of semi-linear evolution equations," J. Diff. Eq. **43** 73-92 (1982)

and, for a recent pedagogical discussion,

R. Temam, "Suitable initial conditions," J. Comp. Phys. **218** 443-450 (2006).

These papers discuss not only the incompressible Navier-Stokes equation but also other similar parabolic problems, such as the linear heat equation. In general, compatibility conditions are required on the initial data, in order to guarantee a given degree of smoothness up to time $t = 0$. The first compatibility condition on $\mathbf{u}_0$ is that, in addition to $\mathbf{u}_0 \in H$, also

$$\mathbf{u}_0 \in H^1(\Omega, \mathbb{R}^d) \text{ and } \mathbf{u}_0 = \mathbf{0} \text{ on } \partial\Omega$$

which is equivalent to requiring that

$$\mathbf{u}_0 \in V.$$

This condition is necessary and sufficient to have a strongly continuous solution, $\mathbf{u} \in C(0, T; V)$. To give even more smoothness, the next-order condition or so-called second compatibility condition is that

$$\left.\frac{d\mathbf{u}}{dt}\right|_{t=0} \in V,$$

or equivalently that

$$\mathbf{B}(\mathbf{u}_0, \mathbf{u}_0) + \nu A\mathbf{u}_0 - \mathbf{f}_0^B \in V.$$

In order to express this condition more transparently, we shall use the definition of Leray projection in terms of the Helmholtz decomposition to write

$$\mathbf{B}(\mathbf{u}_0, \mathbf{u}_0) + \nu A\mathbf{u}_0 = (\mathbf{u}_0 \cdot \boldsymbol{\nabla})\mathbf{u}_0 - \nu\triangle\mathbf{u}_0 - \boldsymbol{\nabla} p_0 \qquad (\star)$$

where p_0 is the solution of the Newmann problem

$$\left\{ \begin{array}{rcl} \triangle p_0 & = & \frac{\partial u_{oi}}{\partial x_j}\frac{\partial u_{oj}}{\partial x_i} \\ \frac{\partial p_0}{\partial n} & = & \mathbf{n} \cdot (\nu\triangle\mathbf{u}_0) \end{array} \right. \qquad (*)$$

We have assumed, for simplicity, that $\mathbf{f}_0^B \in V$ and already satisfies the stick condition at $\partial\Omega$. Because of the choise of p_0, both sides of $(\star)$ are divergence-free and have zero normal component at the boundary $\partial\Omega$. Thus, the second compatibility condition that $\mathbf{B}(\mathbf{u}_0, \mathbf{u}_0) + \nu A\mathbf{u}_0 \in V$

reduces to the condition that also the tangential component must vanish at the boundary, i.e. that

$$\mathbf{n} \times [(\mathbf{u}_0 \cdot \boldsymbol{\nabla})\mathbf{u}_0 - \nu\triangle\mathbf{u}_0 - \boldsymbol{\nabla} p_0] = 0 \text{ on } \partial\Omega$$

or, since $\mathbf{u}_0 = 0$ on $\partial\Omega$,

$$\mathbf{n} \times [\nu\triangle\mathbf{u}_0 + \boldsymbol{\nabla} p_0] = 0 \text{ on } \partial\Omega.$$

This is the simplest, most explicit form of the second compatibility condition for the Navier-Stokes equation.

The above discussion implicitly answers the question about the proper boundary conditions for pressure in the Navier-Stokes equation. We may apply the above arguments to any time $t > 0$, with $\mathbf{u}(t)$ then playing the role of the initial condition $\mathbf{u}_0$. The introduction of the pressure by the Leray projector implies that we may take p to solve the Neumann problem

$$\begin{cases} \triangle p_0 &= \frac{\partial u_i}{\partial x_j}\frac{\partial u_o}{\partial x_i} \\ \frac{\partial p}{\partial n} &= \mathbf{n} \cdot (\nu\triangle\mathbf{u}) \end{cases} \qquad (N)$$

However, if the Navier-Stokes solution is smooth in the region $\Omega \times [t, T]$, then the 2nd compatibility condition implies that also

$$\mathbf{n} \times [\nu\triangle\mathbf{u} + \boldsymbol{\nabla} p] = 0 \text{ on } \partial\Omega \qquad (T).$$

From this we can see that p satisfies also a suitable Dirichlet problem obtained by integrating the above equation around the boundary. In general, the solution of these two elliptic problems for p — the Neumann problem and the Dirichlet problem — would not have to coincide, for an arbitrarily selected $\mathbf{u}$ and p. However, here $(\mathbf{u}, p)$ are in a special relationship, as a smooth solution of the Navier-Stokes equation. In this case, compatibility requires that the Neumann and Dirichlet pressures are exactly the same!

We emphasize that these facts have nothing to do with the possible occurence of singularities in finite time for the 3D Navier-Stokes equation. Smooth solutions of the type considered (up to the boundary and the initial time t) are guaranteed to exist in 2D and in 3D at low Reynolds numbers. Even in 3D at high Reynolds number, the singularities — if any exist — are confined to a zero-measure set, at most 1-dimensional in spacetime. Away from this singular set, the Navier-Stokes solution is smooth up to the boundary $\partial\Omega$ of the domain and at each positive time $t > 0$. The data $\mathbf{u}(t)$ has been "prepared" by the prior evolution so that all the compatibility conditions are automatically satisfied at time $t > 0$, including the condition (T) on the tangential pressure gradient at the boundary. As we shall see next, this condition has, in fact, an important physical significance, as it is directly related to the generation of vorticity at the boundary.

Vorticity dynamics in domains with boundaries

Given a solution of the Navier-Stokes equation $(\mathbf{u}, p)$, we know that the curl $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ will satisfy the Helmholtz vorticity equation

$$(\partial_t + \mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} + \nu\triangle\boldsymbol{\omega} + \boldsymbol{\nabla} \times \mathbf{f}^B$$

or

$$\partial_t\boldsymbol{\omega} = \boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega} - \nu\boldsymbol{\nabla} \times \boldsymbol{\omega} + \mathbf{f}^B)$$

which may be written in conservation form as

$$\partial_t\omega_i + \partial_j\Sigma_{ji} = 0$$

with

$$\Sigma_{ij} = u_i\omega_j - u_j\omega_i + \nu(\tfrac{\partial\omega_i}{\partial x_j} - \tfrac{\partial\omega_j}{\partial x_i}) + \epsilon_{ijk}f_k^B.$$

If we integrate the vorticity conservation law over the flow domain, then we obtain by the divergence theorem

$$\tfrac{d}{dt}\int_\Omega \omega_i d^3x = \int_{\partial\Omega} \Sigma_{ij}n_j dA$$

where $\mathbf{n}$ is the outward-pointing unit normal. This suggests that we can interpret

$$\boldsymbol{\sigma}_i \equiv \Sigma_{ij}n_j$$

as a local vorticity source density at the wall. Because of the antisymmetry of Σ_{ij}, we see that

$$\mathbf{n} \cdot \boldsymbol{\sigma} = 0$$

consistent with our earlier conclusion that, for stick b.c., $\mathbf{n}\cdot\boldsymbol{\omega} = 0$. If we assume that $\mathbf{n}\times\mathbf{f}^B = 0$, then we can simplify the above expression as

$$\sigma_i = \nu(\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i})n_j$$

using both $\mathbf{n} \cdot \mathbf{u} = 0$ and $\mathbf{n} \cdot \boldsymbol{\omega} = 0$, or

$$\boldsymbol{\sigma} = -\mathbf{n} \times \nu(\boldsymbol{\nabla} \times \boldsymbol{\omega}).$$

It is sometimes preferable to use instead the inward - pointing unit normal $\mathbf{n}' = -\mathbf{n}$, so that instead

$$\boldsymbol{\sigma} = \mathbf{n}' \times \nu(\boldsymbol{\nabla} \times \boldsymbol{\omega}).$$

The above argument is not entirely satisfactory, since we obtained only an integral formula

$$\frac{d}{dt} \int_{\Omega} \omega_i d^3x = \int_{\partial\Omega} \sigma_i dA$$

and this does not permit an unambiguous identification of $\boldsymbol{\sigma}$ as a local vorticity source. A better argument may be based on the Kelvin circulation theorem. Let us consider any direction $\mathbf{e}_i$ locally tangent to the surface, $\mathbf{n} \cdot \mathbf{e}_i = 0$, and consider the small square loop C normal to $\mathbf{e}_i$, with one edge lying in the boundary surface $\partial\Omega$:

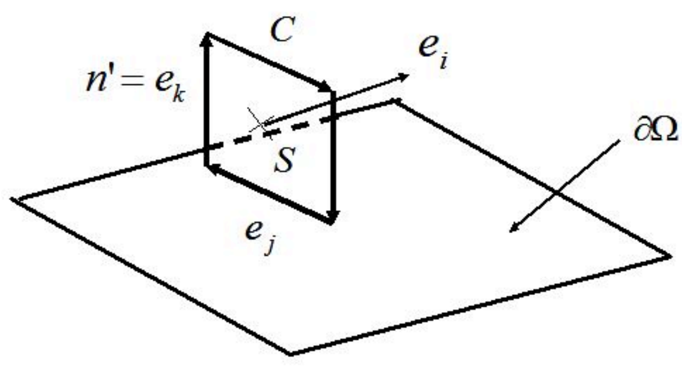


Figure 1.

Here, $\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k = \mathbf{n}'$ and the edge parallel to $\mathbf{e}_j$ lies in the boundary surface $\partial\Omega$.

Then,

$$\int_S \omega_i dx_j dx_k = \oint_C \mathbf{u} \cdot d\mathbf{x}$$

and, if we let $S(t)$ and $C(t)$ be the surface and loop advected by the fluid, then

$$\frac{d}{dt} \int_{S(t)} \omega_i dx_j dx_k |_{t=0} = \frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x}|_{t=0} = -\nu \oint_C (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\mathbf{x}$$

We see that the bottom segment parallel to $\mathbf{e}_j$ is stationary in the surface (because of the stick conditions) and that it contributes a term

$$-\nu \int_{C\cap\partial\Omega} (\boldsymbol{\nabla}\times\boldsymbol{\omega})_j dx_j$$

to the time-derivative of the flux $\int_S \omega_i dA$. We therefore identify

$$\sigma_i = -\nu(\boldsymbol{\nabla}\times\boldsymbol{\omega})_j$$

or, for any direction,

$$\boldsymbol{\sigma} = \mathbf{n}' \times \nu(\boldsymbol{\nabla}\times\boldsymbol{\omega})$$

in agreement with our earlier finding. The above argument also reveals the precise meaning of the "vorticity source density" $\boldsymbol{\sigma}$. If one considers <u>any</u> loop C with part of its length in the boundary $\partial\Omega$, then the contribution of the boundary segment to the time-derivative of the vorticity flux through C is given by

$$\int_{C\cap\partial\Omega} \boldsymbol{\sigma}\cdot(\mathbf{t}\times\mathbf{n}')ds$$

where $\mathbf{t}$ is the unit tangent vector along the curve C, for a specified orientation. With this convention the flux is measured in the direction to the righthand side of the curve C as one moves along it, i.e. $||$ to $\mathbf{t}\times\mathbf{n}'$.

The expression that we have derived for the "vorticity source density"

$$\boldsymbol{\sigma} = \mathbf{n}' \times \nu(\boldsymbol{\nabla}\times\boldsymbol{\omega})$$

is the same as that obtained by

> F. A. Lyman, "Vorticity production at a solid boundary," Appl. Mech. Rev. **43** 157-158 (1990)

These seems, however, to be some controversy about this result! It has been argued by

> J. Z. Wu and J. M. Wu, "Interaction between a solid surface and a viscous compressible flow field," J. Fluid Mech. **254** 183-211 (1993); "Vorticity dynamics on boundaries," Adv. in Appl. Mech. **32** 119-275 (1996)

that Lyman's prescription (and ours!) is incorrect and they suggest instead that

$$\boldsymbol{\sigma}_{\rm Wu} = \nu \frac{\partial \boldsymbol{\omega}}{\partial n} = \nu(\mathbf{n} \cdot \boldsymbol{\nabla})\boldsymbol{\omega}$$

with outward-pointing normal $\mathbf{n}$. However, this prescription seems to us based on fallacious arguments, whereas our derivation from the Kelvin Theorem is particularly straightforward. We note furthermore that $\boldsymbol{\sigma}$ ought to lie parallel to vorticity vectors generated within the boundary surface and thus satisfy $\mathbf{n}\times\boldsymbol{\sigma} = 0$. While Lyman's prescription satisfies this condition, $\mathbf{n} \times \boldsymbol{\sigma}_{\rm Wu} \neq 0$ in general!

Further insight and simplification can be obtained if one uses the Navier-Stokes momentum balance

$$D_t\mathbf{u} = -\boldsymbol{\nabla} p - \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}$$

which can be solved to give

$$-\nu\boldsymbol{\nabla} \times \boldsymbol{\omega} = \boldsymbol{\nabla} p + D_t\mathbf{u}$$

and thus

$$\boldsymbol{\sigma} = -\mathbf{n} \times \nu(\boldsymbol{\nabla} \times \boldsymbol{\omega}) = \mathbf{n} \times (\boldsymbol{\nabla} p + D_t\mathbf{u})$$

This form of the vorticity source density was first derived by

M. J. Lighthill, "Introduction: Boundary Layer Theory" in Laminar Boundary Theory, L. Rosenhead, ed. (Oxford University Press, Oxford, 1963), pp.46-113.

B. R. Morton, "The generation and decay of vorticity," Geophys. Astrophys. Fluid Dyn. **28** 277-308 (1984)

who obtained the terms from the tangential pressure gradient and the tangential boundary acceleration, respectively. Note that the above derivation was based on precisely the 2nd compatibility relation (T) for the Navier-Stokes solution, generalized to the case of a moving boundary. This result reveals that the 2nd compatibility condition has a very fundamental significance related to the generation of vorticity at a boundary. The term identified by Morton, $\mathbf{n} \times (D_t\mathbf{u})$, contributes only if the boundary is accelerating, either continuously or impulsively. If we restrict our attention to uniformly moving (or stationary) boundaries, then we recover the original Lighthill (1963) result

$$\boldsymbol{\sigma} = \mathbf{n} \times (\boldsymbol{\nabla} p) = -\mathbf{n}' \times (\boldsymbol{\nabla} p).$$

This formula has several remarkable implications. First, one can see that the generation of vorticity is essentially inviscid and does not vanish in the limit as $\nu \to 0$. Second, the generated vortex lines on the surface $\partial\Omega$ ought to tend to be parallel to the isobars, or lines of constant pressure p. E.g. the following plot from the paper

> J.-G. Liu, J. Liu, and R. L. Pego, "Stable and accurate pressure approximation for unsteady incompressible flow," J. Comp. Phys. **229** 3428–3453 (2010)

shows vortex lines and pressure contours at the wall for a driven cavity flow:

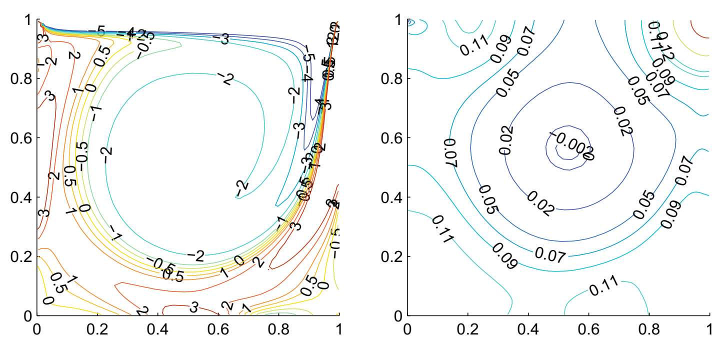


Fig. 8. Driven cavity, $\nu = 1/1000$. P1/P1 with 8192 P1 elements (dof = 4225) for each variable. $h_{\min} = 0.00594$, $h_{\max} = 0.0397$, $\Delta t = 0.006$, $T = 50$. Top: (3,2) SC scheme. Bottom: (3,2) PA scheme. From left to right: vorticity contour plots, pressure contour plots.

Although the vortex source vector $\boldsymbol{\sigma}$ is instantaneously tangent to the pressure isolines, the vorticity vector $\boldsymbol{\omega}$ is a dynamical quantity which includes contributions both from the instantaneous source and from the evolution of past values.

For a review of these issues with many references to the literature, see

> J. Z. Wu & J. M. Wu, "Boundary vorticity dynamics since Lighthill's 1963 article: review and development," Theoret. Comput. Fluid Dyn. **10** 459-474 (1998).

Of course, these authors defend the prescription $\boldsymbol{\sigma}_{\rm Wu}$!!!

A final remark is that the Constantin-Iyer stochastic formulation of incompressible Navier-Stokes has been recently generalized to domains with boundaries, in

P. Constantin & G. Iyer, "A stochastic-Lagrangian approach to the Navier-Stokes equations in domains with boundary," Ann. Appl. Probab. **21** 1466–1492 (2011)

and yields some new insight into the generation of vorticity at walls. The major difference is that now stochastic Lagrangian trajectories moving backward in time can hit the domain boundary ∂D before reaching the initial time $t_0 = 0$, as illustrated here:

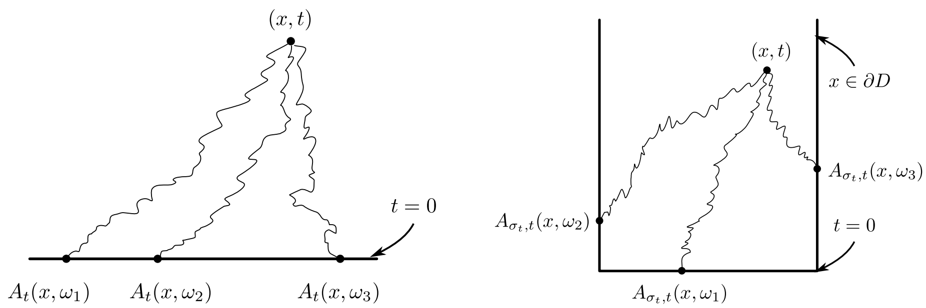


FIG. 1. *Three sample realizations of A without boundaries* (left) *and with boundaries* (right).

In the case without boundaries, the Constantin-Iyer representation can be stated as

$$\mathbf{u}(\mathbf{x},t) = P\overline{\boldsymbol{\nabla}_x \tilde{\mathbf{A}}_t^0(\mathbf{x}) \cdot \mathbf{u}_0 \left(\tilde{\mathbf{A}}_t^0(\mathbf{x})\right)} \qquad (CI)$$

where the stochastic trajectories solve

$$d\tilde{\mathbf{A}}_t^s = \mathbf{u}(\tilde{\mathbf{A}}_t^s, s)ds + \sqrt{2\nu}\, d\tilde{\mathbf{W}}(s),\ \ s < t; \qquad \tilde{\mathbf{A}}_t^t = \mathbf{x},$$

the overline $\overline{()}$ is average over the Brownian motions, and P is the Leray projection. Integrating the variable $\mathbf{x}$ in (CI) over a loop C yields the stochastic Kelvin theorem previously discussed. In the presence of a boundary this formula is modified as follows:

$$\mathbf{u}(\mathbf{x},t) = P\overline{\boldsymbol{\nabla}_x \tilde{\mathbf{A}}_t^{\tau_t(\mathbf{x})}(\mathbf{x}) \cdot \mathbf{w} \left(\tilde{\mathbf{A}}_t^{\tau_t(\mathbf{x})}(\mathbf{x}), \tau_t(\mathbf{x})\right)} \qquad (CI*)$$

where $\tau_t(\mathbf{x})$ is the *backward exit time*, or the time at which the stochastic trajectory leaving $\mathbf{x}$ at time t reaches the space-time boundary. The space-time boundary value $\mathbf{w}$ is

$$\mathbf{w} = \mathbf{u}_0 \text{ on } D \times \{0\},$$

the same as before, while

$$\mathbf{w} = \boldsymbol{\nabla}\varphi \text{ on } \partial D \times [0, t],$$

for a scalar function which solves the equation

$$\partial_t \varphi + (\mathbf{u} \cdot \boldsymbol{\nabla})\varphi - \nu \triangle \varphi + \frac{1}{2}|\mathbf{u}|^2 - p = c(t), \quad \varphi(0) = c_0.$$

It is interesting that any spatial constants $c(t)$ and c_0 can be employed in this equation, and also any boundary conditions for φ on ∂D. Any such choice yields the same pressure gradient $\boldsymbol{\nabla}p$ and, substituted into $(CL*)$, yields the Navier-Stokes solution with "stick" or zero-Dirichlet boundary conditions ($\mathbf{u} = \mathbf{0}$) at the wall! This representation is a generalization to incompressible Navier-Stokes equation in a bounded domain of the Kuzmin-Oseledets formulation of incompressible Euler equation. See

G. A. Kuzmin, "Ideal incompressible hydrodynamics in terms of the vortex momentum density," Phys. Lett. A **96** 88–90 (1983)

V. I. Oseledets, "On a new way of writing the Navier-Stokes equation. The Hamiltonian formalism," Commun. Moscow Math. Soc (1988); Russ. Math. Surveys **44** 210–211 (1988).

In this formalism, the field $\mathbf{p}(\mathbf{x}, t)$ defined by $(CI*)$ without the Leray projection P is termed the "vortex momentum density" and $\mathbf{u} = \mathbf{p} - \boldsymbol{\nabla}\varphi = P\mathbf{p}$.

Some remarks on numerical boundary conditions

Although the issue of boundary conditions for the continuum Navier-Stokes equation seems settled, many interesting and important issues remain for numerical solution methods. For example, see the article of Rempfer (2006) and Liu, Liu & Pego (201) for references to the literature. Some common methods, such as the projection method, do not approximate the pressure well near the wall in a boundary layer. See

R. Temam, "Remark on the pressure boundary condition for the projection method," Theoret. Comput. Fluid Dyn. **3** 181-184 (1991)

There are also important issues about the boundary conditions for the Helmholtz equation in the vorticity formulation of the Navier-Stokes equation. For this, see

> J.-Z. Wu, H.-Y. Ma and M. D. Zhou, Vorticity and Vortex Dynamics (Springer, Berlin, 2006), Section 4.5 "Vorticity-Based Formulation of Viscous Flows Problem"
>
> J. -Z. Wu et al. "Dynamic vorticity condition: theoretical analysis and numerical implementation," Int. J. Num. Math. Fluids **19** 905-938 (1994)

The difficulty is that the proper physical boundary conditions are the stick conditions on the velocity:

$$\mathbf{u}|_{\partial\Omega} = 0$$

and the velocity is nonlocally related to the vorticity $\boldsymbol{\omega}$. Given the vorticity satisfying $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$, one must solve the problem

$$\boldsymbol{\nabla}\times\mathbf{u} = \boldsymbol{\omega},\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0,\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}$$

to determine $\mathbf{u}$. This corresponds to the elliptic Poisson problem $-\triangle\mathbf{u} = \boldsymbol{\nabla}\times\boldsymbol{\omega}$ with Dirichlet b.c. The boundary conditions for $\boldsymbol{\omega}$ are inherently nonlocal and global and, without some approximation, cannot be formulated as standard local conditions.

Finally, we remark that the Constantin-Iyer stochastic representation of the viscous Helmholtz equation

$$\partial_t\boldsymbol{\omega} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} + \nu\triangle\boldsymbol{\omega}$$

can be easily generalized to domains with boundaries, by taking

$$\boldsymbol{\omega}(\mathbf{x},t) = \overline{(\boldsymbol{\nabla}\mathbf{A}_t^{\tau_t(\mathbf{x})}(\mathbf{x}))^{\top-1}\tilde{\boldsymbol{\omega}}(\mathbf{A}_t^{\tau_t(\mathbf{x})}(\mathbf{x}),\tau_t(\mathbf{x}))}$$

where the function $\tilde{\boldsymbol{\omega}}$ is defined simply to coincide with $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ on the space-time boundary $(D\times\{t=0\})\cup(\partial D\times\{t>0\})$. This representation thus has the same practical problem as does the standard PDE formulation of constructing the proper boundary values on the vorticity to correspond to stick conditions on the velocity.